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**Sheldon Axler
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Harmonic Function Theory



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Harmonic Function Theory

With 16 Illustrations



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Preface

Harmonic functions—the solutions of Laplace’s equation—play a crucial role in many areas of mathematics, physics, and engineering. But learning about them is not always easy. At times each of the authors has agreed with Lord Kelvin and Peter Tait, who wrote ([12], Preface)

There can be but one opinion as to the beauty and utility of this analysis of Laplace; but the manner in which it has been hitherto presented has seemed repulsive to the ablest mathematicians, and difficult to ordinary mathematical students.

The quotation has been included mostly for the sake of amusement, but it does convey a sense of the difficulties the uninitiated sometimes encounter.

The main purpose of our text, then, is to make learning about harmonic functions easier. The only prerequisite for the book is a solid foundation in real and complex analysis, together with some basic results from functional analysis. The first fifteen chapters of Rudin’s *Real and Complex Analysis*, for example, provide sufficient preparation.

In several cases we simplify standard proofs. For example, we replace the usual tedious calculations showing that the Kelvin transform of a harmonic function is harmonic with some straightforward observations that we believe are more revealing. Another example is

our proof of Bôcher's Theorem, which is more elementary than the classical proofs.

We also present material not usually covered in standard treatments of harmonic functions. The section on the Schwarz Lemma and the chapter on Bergman spaces are examples. For completeness, we include some topics in analysis that frequently slip through the cracks in a beginning graduate student's curriculum, such as real-analytic functions.

We rarely attempt to trace the history of the ideas presented in this book. Thus the absence of a reference does not imply originality on our part.

In addition to writing the text, the authors have developed a software package to manipulate many of the expressions that arise in harmonic function theory. Our software package, which uses many results from this book, can perform symbolic calculations that would take a prohibitive amount of time if done without a computer. For example, the Poisson integral of any polynomial can be computed exactly. Appendix B explains how readers can obtain our software package free of charge.

This book has its roots in a graduate course at Michigan State University taught by one of the authors and attended by the other authors along with a number of graduate students. The topic of harmonic functions was presented with the intention of moving on to different material after introducing the basic concepts. We did not move on to different material. Instead, we began to ask natural questions about harmonic functions. Lively and illuminating discussions ensued. A freewheeling approach to the course developed; answers to questions someone had raised in class or in the hallway were worked out and then presented in class (or in the hallway). Discovering mathematics in this way was a thoroughly enjoyable experience. We will consider this book a success if some of that enjoyment shines through in these pages.

Acknowledgments

Our book has been improved by our students. We take this opportunity to thank them for catching errors and making suggestions while attending courses at Michigan State University based on material in this book.

Among the many mathematicians who have influenced our outlook on harmonic function theory, we give special thanks to Dan Luecking for helping us to better understand Bergman spaces, and to Elias Stein and Guido Weiss for their book [10], which contributed greatly to our knowledge of spherical harmonics.

Lastly we thank the typists, who labored endlessly on this project. Although they produced some of the worst typing we have seen, the number of errors from one draft to the next did, on occasion, actually decrease. The typists were: Sheldon Axler, Paul Bourdon, and Wade Ramey.

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CHAPTER 1

Basic Properties of Harmonic Functions

Definitions and Examples

Harmonic functions, for us, live on open subsets of real Euclidean spaces. Throughout this book, n will denote a fixed positive integer greater than 1 and Ω will denote an open, non-empty subset of $\mathbf{R}^n$. A twice continuously differentiable, complex-valued function u defined on Ω is *harmonic* on Ω if

$$\Delta u \equiv 0,$$

where $\Delta = D_1^2 + \cdots + D_n^2$ and D_j^2 denotes the second partial derivative with respect to the j^{th} coordinate variable. The operator Δ is called the *Laplacian*, and the equation $\Delta u \equiv 0$ is called *Laplace's equation*. We say that a function u defined on a (not necessarily open) set $E \subset \mathbf{R}^n$ is harmonic on E if u can be extended to a function harmonic on an open set containing E .

We let $x = (x_1, \dots, x_n)$ denote a typical point in $\mathbf{R}^n$ and let $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$ denote the Euclidean norm of x .

The simplest nonconstant harmonic functions are the coordinate functions; for example, $u(x) = x_1$. A slightly more complex example is the function on $\mathbf{R}^3$ defined by

$$u(x) = x_1^2 + x_2^2 - 2x_3^2 + ix_2.$$

As we will see later, the function

$$u(x) = |x|^{2-n}$$

is vital to harmonic function theory when $n > 2$; the reader should verify that this function is harmonic on $\mathbf{R}^n \setminus \{0\}$.

We can obtain additional examples of harmonic functions by differentiation, noting that for smooth functions the Laplacian commutes with any partial derivative. In particular, differentiating the last example with respect to x_1 shows that $x_1|x|^{-n}$ is harmonic on $\mathbf{R}^n \setminus \{0\}$ when $n > 2$. (We will soon prove that every harmonic function is infinitely differentiable; thus any partial derivative of a harmonic function is harmonic.)

The function $x_1|x|^{-n}$ is harmonic on $\mathbf{R}^n \setminus \{0\}$ even when $n = 2$. This can be verified directly or by noting that $x_1|x|^{-2}$ is a partial derivative of $\log|x|$, a harmonic function on $\mathbf{R}^2 \setminus \{0\}$. The function $\log|x|$ plays the same role when $n = 2$ that $|x|^{2-n}$ plays when $n > 2$. Notice that as $|x| \rightarrow \infty$, $\log|x| \rightarrow \infty$ whereas $|x|^{2-n} \rightarrow 0$; note also that $\log|x|$ is neither bounded above nor below, while $|x|^{2-n}$ is always positive. These facts hint at the contrast between harmonic function theory in the plane and in higher dimensions. Another difference arises from the connection between holomorphic and harmonic functions in the plane: A real-valued function on $\Omega \subset \mathbf{R}^2$ is harmonic if and only if it is locally the real part of a holomorphic function. No comparable result exists in higher dimensions.

Invariance Properties

Throughout this book, all functions are assumed to be complex valued unless stated otherwise. For k a positive integer, let $C^k(\Omega)$ denote the set of k times continuously differentiable functions on Ω ; $C^\infty(\Omega)$ is the set of functions that belong to $C^k(\Omega)$ for every k . For $E \subset \mathbf{R}^n$, we let $C(E)$ denote the set of continuous functions on E .

Because the Laplacian is linear on $C^2(\Omega)$, sums and scalar multiples of harmonic functions are harmonic.

For $y \in \mathbf{R}^n$ and u a function on Ω , the y -translate of u is the function on $\Omega + y$ whose value at x is $u(x - y)$. Clearly translations of harmonic functions are harmonic.

For a positive number r and u a function on Ω , the r -dilate

of u , denoted u_r , is the function

$$(u_r)(x) = u(rx)$$

defined for x in $(1/r)\Omega = \{(1/r)w : w \in \Omega\}$. If $u \in C^2(\Omega)$, then a simple computation shows

$$\Delta(u_r) = r^2(\Delta u)_r$$

on $(1/r)\Omega$. Hence dilates of harmonic functions are harmonic.

The reader may have noticed the formal similarity between the Laplacian $\Delta = D_1^2 + \cdots + D_n^2$ and the function $|x|^2 = x_1^2 + \cdots + x_n^2$, whose level sets are spheres centered at the origin. The connection between harmonic functions and spheres is central to harmonic function theory. The mean-value property, which we discuss in the next section, best illustrates this connection. Another connection involves linear transformations on $\mathbf{R}^n$ that preserve the unit sphere; such transformations are called *orthogonal*. A linear map $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$ is orthogonal if and only if $|Tx| = |x|$ for all $x \in \mathbf{R}^n$. Simple linear algebra shows that T is orthogonal if and only if the column vectors of the matrix of T (with respect to the standard basis for $\mathbf{R}^n$) form an orthonormal set.

We now show that the Laplacian commutes with orthogonal transformations; more precisely, if T is orthogonal and $u \in C^2(\Omega)$, then

$$\Delta(u \circ T) = (\Delta u) \circ T$$

on $T^{-1}(\Omega)$. To prove this, let $[t_{jk}]$ denote the matrix of T relative to the standard basis for $\mathbf{R}^n$. Then

$$D_m(u \circ T) = \sum_{j=1}^n t_{jm}(D_j u) \circ T,$$

where D_m denotes the partial derivative with respect to the m^{th} coordinate variable. Differentiating once more and summing over m yields

$$\begin{aligned}
\Delta(u \circ T) &= \sum_{m=1}^n \sum_{j,k=1}^n t_{km} t_{jm} (D_k D_j u) \circ T \\
&= \sum_{j,k=1}^n \left(\sum_{m=1}^n t_{km} t_{jm} \right) (D_k D_j u) \circ T \\
&= \sum_{j=1}^n (D_j D_j u) \circ T \\
&= (\Delta u) \circ T,
\end{aligned}$$

as desired. The function $u \circ T$ is called a *rotation* of u . The preceding calculation shows that rotations of harmonic functions are harmonic.

The Mean-Value Property

Many basic properties of harmonic functions follow from Green's identity (which we will need mainly in the special case when Ω is a ball):

$$\int_{\Omega} (u \Delta v - v \Delta u) dV = \int_{\partial\Omega} (u D_{\mathbf{n}} v - v D_{\mathbf{n}} u) ds.$$

Here Ω is a bounded open subset of $\mathbf{R}^n$ with smooth boundary, and u and v are C^2 -functions on a neighborhood of $\bar{\Omega}$, the closure of Ω . The measure $V = V_n$ is Lebesgue volume measure on $\mathbf{R}^n$, and s denotes surface-area measure on $\partial\Omega$ (see Appendix A for a discussion of integration over balls and spheres). The symbol $D_{\mathbf{n}}$ denotes differentiation with respect to the outward unit normal $\mathbf{n}$. Thus for $\zeta \in \partial\Omega$, $(D_{\mathbf{n}} u)(\zeta) = (\nabla u)(\zeta) \cdot \mathbf{n}(\zeta)$, where $\nabla u = (D_1 u, \dots, D_n u)$ denotes the gradient of u and $\cdot$ denotes the usual Euclidean inner product.

Green's identity follows easily from the familiar divergence theorem of advanced calculus:

$$\int_{\Omega} \operatorname{div} \mathbf{w} dV = \int_{\partial\Omega} \mathbf{w} \cdot \mathbf{n} ds.$$

Here $\mathbf{w} = (w_1, \dots, w_n)$ is a smooth vector field (a $\mathbf{C}^n$ -valued function whose components are continuously differentiable) on a neighborhood of $\bar{\Omega}$, and $\operatorname{div} \mathbf{w}$, the divergence of $\mathbf{w}$, is defined to be

$D_1 w_1 + \cdots + D_n w_n$. To obtain Green's identity from the divergence theorem, simply let $\mathbf{w} = u \nabla v - v \nabla u$ and compute.

The following useful form of Green's identity occurs when u is harmonic and $v \equiv 1$:

$$1.1 \quad \int_{\partial\Omega} D_{\mathbf{n}} u \, ds = 0.$$

Green's identity is the key to the proof of the mean-value property. Before stating the mean-value property, we introduce some notation: $B(a, r) = \{x \in \mathbf{R}^n : |x - a| < r\}$ is the open ball centered at a of radius r ; $\bar{B}(a, r)$ is the closed ball centered at a of radius r ; the unit ball $B(0, 1)$ is denoted by B and its closure by $\bar{B}$. When the dimension is important we write B_n in place of B . The unit sphere, the boundary of B , is denoted by S ; normalized surface-area measure on S is denoted by σ (so that $\sigma(S) = 1$). The measure σ is the unique Borel probability measure on S that is rotation invariant (meaning $\sigma(T(E)) = \sigma(E)$ for every Borel set $E \subset S$ and every orthogonal transformation T).

1.2 The Mean-Value Property: *If u is harmonic on $\bar{B}(a, r)$, then $u(a)$ equals the average of u over $\partial B(a, r)$. More precisely,*

$$u(a) = \int_S u(a + r\zeta) \, d\sigma(\zeta).$$

PROOF: First assume that $n > 2$. Without loss of generality we may assume that $B(a, r) = B$. Fix $\varepsilon \in (0, 1)$. Apply Green's identity with $\Omega = \{x \in \mathbf{R}^n : \varepsilon < |x| < 1\}$ and $v(x) = |x|^{2-n}$ to obtain

$$\begin{aligned} 0 &= (2-n) \int_S u \, ds - (2-n)\varepsilon^{1-n} \int_{\varepsilon S} u \, ds \\ &\quad - \int_S D_{\mathbf{n}} u \, ds - \varepsilon^{2-n} \int_{\varepsilon S} D_{\mathbf{n}} u \, ds. \end{aligned}$$

By 1.1, the last two terms are 0, thus

$$\int_S u \, ds = \varepsilon^{1-n} \int_{\varepsilon S} u \, ds,$$

which is the same as

$$\int_S u \, d\sigma = \int_S u(\varepsilon\zeta) \, d\sigma(\zeta).$$

Letting $\varepsilon \rightarrow 0$ and using the continuity of u at 0, we obtain the desired result.

The proof when $n = 2$ is the same, except that $|x|^{2-n}$ should be replaced by $\log |x|$. $\square$

Harmonic functions also have a mean-value property with respect to volume measure. The polar coordinates formula for integration on $\mathbf{R}^n$ is indispensable here. The formula states that for a Borel measurable, integrable function f on $\mathbf{R}^n$,

$$\frac{1}{nV(B)} \int_{\mathbf{R}^n} f \, dV = \int_0^\infty r^{n-1} \int_S f(r\zeta) \, d\sigma(\zeta) \, dr$$

(see [9], Chapter 8, Exercise 6). The constant $nV(B)$ arises from the normalization of σ (choosing f to be the characteristic function of B shows that $nV(B)$ is the correct constant).

1.3 The Mean-Value Property, Volume Version: *If u is harmonic on $\bar{B}(a, r)$, then $u(a)$ equals the average of u over $B(a, r)$. More precisely,*

$$u(a) = \frac{1}{V(B(a, r))} \int_{B(a, r)} u \, dV.$$

PROOF: We can assume that $B(a, r) = B$. Apply the polar coordinates formula discussed above with f equal to u times the characteristic function of B , and then use the spherical mean-value property (Theorem 1.2). $\square$

We will see later that the mean-value property characterizes harmonic functions.

The Maximum Principle

An important consequence of the mean-value property is the following maximum principle for harmonic functions.

1.4 Theorem: *Let Ω be connected, and let u be harmonic and real valued on Ω . If u has either a maximum or a minimum in Ω , then u is constant.*

PROOF: Suppose u attains a maximum at $a \in \Omega$. Choose $r > 0$ such that $\bar{B}(a, r) \subset \Omega$. If u were less than $u(a)$ at some point of $B(a, r)$, then the continuity of u would show that the average of u over $B(a, r)$ is less than $u(a)$, contradicting 1.3. Therefore u is constant on $B(a, r)$, proving that the set where u attains its maximum is open in Ω . Because this set is also closed in Ω (again by the continuity of u), it must be all of Ω by connectivity. The function u is thus constant on Ω , as desired.

If u attains a minimum in Ω , we can apply this argument to $-u$. $\square$

The following corollary, whose proof immediately follows from the preceding theorem, is frequently useful. (Note that the connectivity of Ω is not needed here.)

1.5 Corollary: *Suppose Ω is bounded and u is a continuous real-valued function on $\bar{\Omega}$ that is harmonic on Ω . Then u attains its maximum and minimum values over $\bar{\Omega}$ on $\partial\Omega$.*

The last corollary shows that on a bounded domain a harmonic function is determined by its boundary values. More precisely, for bounded Ω , if u and v are continuous functions on $\bar{\Omega}$ that are harmonic on Ω , and if $u = v$ on $\partial\Omega$, then $u = v$ on Ω . This can fail on an unbounded domain; for example, the harmonic functions $u(x) = 0$ and $v(x) = x_n$ agree on the boundary of the half-space $\{x \in \mathbf{R}^n : x_n > 0\}$.

The next version of the maximum principle can be applied even when Ω is unbounded or when u is not continuous on $\bar{\Omega}$.

1.6 Corollary: *Let u be a real-valued, harmonic function on Ω , and suppose*

$$\limsup_{k \rightarrow \infty} u(a_k) \leq M$$

for every sequence (a_k) in Ω converging either to a point in $\partial\Omega$ or to ∞ . Then $u \leq M$ on Ω .

REMARKS: In the corollary above, M may be infinite. To say that (a_k) converges to ∞ means that $|a_k| \rightarrow \infty$. The corollary is valid if “lim sup” is replaced by “lim inf” and the inequalities are reversed.

PROOF OF COROLLARY 1.6: Let $M' = \sup\{u(x) : x \in \Omega\}$, and choose a sequence (b_k) in Ω such that $u(b_k) \rightarrow M'$. If (b_k) has a subsequence converging to some point $b \in \Omega$, then $u(b) = M'$, which implies u is constant on the component of Ω containing b by the maximum principle (1.4). Hence in this case there is a sequence (a_k) in Ω converging to a boundary point of Ω or to ∞ on which $u = M'$. If no subsequence of (b_k) converges to a point in Ω , then (b_k) has a subsequence (a_k) converging either to a boundary point of Ω or to ∞ . Thus in all cases we obtain $M' \leq M$. $\square$

Theorem 1.4 and Corollaries 1.5 and 1.6 apply only to real-valued functions. The next corollary is a version of the maximum principle for complex-valued functions.

1.7 Corollary: *Let Ω be connected, and let u be harmonic on Ω . If $|u|$ has a maximum in Ω , then u is constant.*

PROOF: Suppose $|u|$ attains a maximum value of M at some point $a \in \Omega$. Choose $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$ and $\lambda u(a) = M$. Then the real-valued harmonic function $\operatorname{Re} \lambda u$ attains its maximum value M at a ; thus by Theorem 1.4, $\operatorname{Re} \lambda u \equiv M$ on Ω . Because $|\lambda u| = |u| \leq M$, we have $\operatorname{Im} \lambda u \equiv 0$ on Ω . Thus λu , and hence u , is constant on Ω . $\square$

No minimum principle holds for $|u|$ (consider $u(x) = x_1$ on B).

Corollary 1.7 is the analogue of Theorem 1.4 for complex-valued harmonic functions; the corresponding analogues of Corollaries 1.5 and 1.6 are also valid. We will be able to prove a local version of the maximum principle after we prove that harmonic functions are real analytic (see 1.25).

We conclude this section with an application of the maximum principle. We have seen that a real-valued harmonic function may have an isolated (nonremovable) singularity; for example, $|x|^{2-n}$ has an isolated singularity at 0 if $n > 2$. However, a real-valued harmonic function u cannot have isolated zeros.

1.8 Corollary: *The zeros of a real-valued harmonic function are never isolated.*

PROOF: Assume u is harmonic and real valued on Ω , $a \in \Omega$, and $u(a) = 0$. Suppose $\bar{B}(a, r) \subset \Omega$. If $u \equiv 0$ on $B(a, r)$, then

we are done. Otherwise u takes on positive and negative values on $\partial B(a, r)$ by the maximum principle. This implies that the connected set $u(\partial B(a, r)) \subset \mathbf{R}$ contains 0. Thus u has a zero on $\partial B(a, r)$ for every small $r > 0$, proving that a is not an isolated zero. $\square$

The hypothesis that u is real valued is needed in the preceding corollary. This is no surprise when $n = 2$; any nonconstant holomorphic function having a zero is an example. When $n \geq 2$, the harmonic function

$$(1 - n)x_1^2 + \sum_{k=2}^n x_k^2 + ix_1$$

is an example; it vanishes only at the origin.

The Poisson Kernel for the Ball

The mean-value property shows that if u is harmonic on $\bar{B}$, then

$$u(0) = \int_S u(\zeta) d\sigma(\zeta).$$

We now show that for any $x \in B$, $u(x)$ is a weighted average of u over S . More precisely, we will show there exists a function P on $B \times S$ such that

$$u(x) = \int_S P(x, \zeta) u(\zeta) d\sigma(\zeta)$$

for every $x \in B$ and every u harmonic on $\bar{B}$.

To discover what P might be, we start with the special case $n = 2$. Suppose u is a real-valued harmonic function on the closed unit disk in $\mathbf{R}^2$. Then $u = \operatorname{Re} f$ for some function f holomorphic on a neighborhood of the closed disk (see Exercise 6 of this chapter). Because $u = (f + \bar{f})/2$, the Taylor series expansion of f implies that u has the form

$$u(r\zeta) = \sum_{j=-\infty}^{\infty} a_j r^{|j|} \zeta^j,$$

where $0 \leq r \leq 1$ and $|\zeta| = 1$. In this formula, take $r = 1$, multiply both sides by ζ^{-k} , then integrate over the unit circle to obtain

$$a_k = \int_S u(\zeta) \zeta^{-k} d\sigma(\zeta).$$

Now let x be a point in the open unit disk, and write $x = r\eta$ with $r \in [0, 1)$ and $|\eta| = 1$. Then

$$\begin{aligned} 1.9 \quad u(x) &= u(r\eta) \\ &= \sum_{j=-\infty}^{\infty} \left(\int_S u(\zeta) \zeta^{-j} d\sigma(\zeta) \right) r^{|j|} \eta^j \\ &= \int_S \left(\sum_{j=-\infty}^{\infty} r^{|j|} (\eta \zeta^{-1})^j \right) u(\zeta) d\sigma(\zeta). \end{aligned}$$

Breaking the last sum into two geometric series, we see that

$$u(x) = \int_S \frac{1 - r^2}{|r\eta - \zeta|^2} u(\zeta) d\sigma(\zeta).$$

Thus, letting $P(x, \zeta) = (1 - |x|^2)/|x - \zeta|^2$, we obtain the desired formula for $n = 2$:

$$u(x) = \int_S P(x, \zeta) u(\zeta) d\sigma(\zeta).$$

Unfortunately, nothing as simple as this works in higher dimensions. To find $P(x, \zeta)$ when $n > 2$, we start with a result we call the symmetry lemma, which will be useful in other contexts as well.

1.10 Symmetry Lemma: *For any nonzero x and y in $\mathbf{R}^n$,*

$$||y|^{-1}y - |y|x| = ||x|^{-1}x - |x|y|.$$

PROOF: Square both sides and expand using the inner product. $\square$

To find P for $n > 2$, we try the same approach used in proving the mean-value property. Suppose that u is harmonic on $\bar{B}$. When proving that $u(0)$ is the average of u over S , we applied Green's identity with $v(y) = |y|^{2-n}$; this function is harmonic on $B \setminus \{0\}$, has a singularity at 0, and is constant on S . Now fix a nonzero point $x \in B$. To show that $u(x)$ is a weighted average of u over S , it is

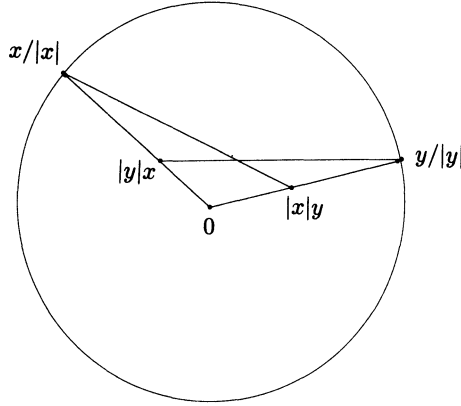


Illustration of the symmetry lemma.

natural this time to try $v(y) = |y - x|^{2-n}$. This function is harmonic on $B \setminus \{x\}$, has a singularity at x , but unfortunately is not constant on S . However, the symmetry lemma (1.10) shows that for $y \in S$,

$$|y - x|^{2-n} = |x|^{2-n} |y - \frac{x}{|x|^2}|^{2-n}.$$

Notice that the right side of this equation is harmonic (as a function of y) on $\bar{B}$. Thus the difference of the left and right sides has all the properties we seek.

So set $v(y) = \mathcal{L}(y) - \mathcal{R}(y)$, where

$$\mathcal{L}(y) = |y - x|^{2-n}, \quad \mathcal{R}(y) = |x|^{2-n} |y - \frac{x}{|x|^2}|^{2-n},$$

and choose ε small enough so that $\bar{B}(x, \varepsilon) \subset B$. Now apply Green's identity much as in the proof of the mean-value property (1.2), with $\Omega = B \setminus \bar{B}(x, \varepsilon)$. We obtain

$$\begin{aligned} 0 &= \int_S u D_{\mathbf{n}} v \, ds - (2-n) s(S) u(x) \\ &\quad - \int_{\partial B(x, \varepsilon)} u D_{\mathbf{n}} \mathcal{R} \, ds - \int_{\partial B(x, \varepsilon)} \mathcal{R} D_{\mathbf{n}} u \, ds \end{aligned}$$

(the mean-value property was used here). Because $u D_{\mathbf{n}} \mathcal{R}$ and $\mathcal{R} D_{\mathbf{n}} u$ are bounded on B , the last two terms approach 0 as $\varepsilon \rightarrow 0$. Hence

$$u(x) = \frac{1}{2-n} \int_S u D_{\mathbf{n}} v \, d\sigma.$$

Setting $P(x, \zeta) = (2-n)^{-1}(D_{\mathbf{n}}v)(\zeta)$, we have the desired formula:

$$1.11 \quad u(x) = \int_S P(x, \zeta) u(\zeta) d\sigma(\zeta).$$

A computation of $D_{\mathbf{n}}v$, which we recommend to the reader (the symmetry lemma may be useful here), yields

$$1.12 \quad P(x, \zeta) = \frac{1 - |x|^2}{|x - \zeta|^n}.$$

The function P derived above is called the *Poisson kernel* for the ball; it plays a key role in the next section.

The Dirichlet Problem for the Ball

We now come to a famous problem in harmonic function theory: Given a continuous function f on S , does there exist a continuous function u on $\bar{B}$, with u harmonic on B , such that $u = f$ on S ? If so, how do we find u ? This is the *Dirichlet problem* for the ball. (Recall that by the maximum principle, if a solution exists, then it is unique.)

We take our cue from the last section: If f happens to be the restriction to S of a function u harmonic on $\bar{B}$, then

$$u(x) = \int_S P(x, \zeta) f(\zeta) d\sigma(\zeta)$$

for all $x \in B$. We solve the Dirichlet problem for B by changing our perspective. Starting with a continuous function f on S , we use the last formula to define an extension of f into B that we hope will have the desired properties.

The reader who wishes may regard the material in the last section as motivation. We now start anew, using 1.12 as the definition of $P(x, \zeta)$.

For arbitrary $f \in C(S)$, we define the *Poisson integral* of f , denoted $P[f]$, to be the function on B given by

$$1.13 \quad P[f](x) = \int_S P(x, \zeta) f(\zeta) d\sigma(\zeta).$$

The next theorem shows that the Poisson integral solves the Dirichlet problem for B .

1.14 Solution of the Dirichlet problem for the ball: Suppose f is continuous on S . Define u on $\bar{B}$ by

$$u(x) = \begin{cases} P[f](x) & \text{if } x \in B \\ f(x) & \text{if } x \in S. \end{cases}$$

Then u is continuous on $\bar{B}$ and harmonic on B .

The proof of 1.14 depends on harmonicity and approximate-identity properties of the Poisson kernel given in the following two propositions.

1.15 Proposition: Let $\zeta \in S$. Then $P(\cdot, \zeta)$ is harmonic on $\mathbf{R}^n \setminus \{\zeta\}$.

To prove this proposition, write $P(x, \zeta) = (1 - |x|^2)|x - \zeta|^{-n}$ and then compute the Laplacian of $P(\cdot, \zeta)$ using the product rule

$$\Delta(uv) = u\Delta v + 2\nabla u \cdot \nabla v + v\Delta u,$$

which is valid for all real-valued twice continuously differentiable functions u and v .

1.16 Proposition: The Poisson kernel has the following properties:

- (a) $P(x, \zeta) > 0$ for all $x \in B$ and all $\zeta \in S$;
- (b) $\int_S P(x, \zeta) d\sigma(\zeta) = 1$ for all $x \in B$;
- (c) for every $\eta \in S$ and every $\delta > 0$,

$$\int_{|\zeta - \eta| > \delta} P(x, \zeta) d\sigma(\zeta) \rightarrow 0 \quad \text{as } x \rightarrow \eta.$$

PROOF: Properties (a) and (c) follow immediately from the formula for the Poisson kernel (1.12).

Taking u to be identically 1 in 1.11 gives (b). To prove (b) without using the motivational material in the last section, note that for $x \in B \setminus \{0\}$,

$$\int_S P(x, \zeta) d\sigma(\zeta) = \int_S P(|\zeta|x, \frac{\zeta}{|\zeta|}) d\sigma(\zeta) = \int_S P(|x|\zeta, \frac{x}{|x|}) d\sigma(\zeta),$$

where the last equality follows from the symmetry lemma (1.10). Proposition 1.15 tells us that $P(|x|\zeta, \frac{x}{|x|})$, as a function of ζ , is harmonic on $\bar{B}$. Thus by the mean-value property we have

$$\int_S P(x, \zeta) d\sigma(\zeta) = P(0, \frac{x}{|x|}) = 1,$$

as desired. Clearly (b) also holds for $x = 0$, completing the proof. $\square$

PROOF OF THEOREM 1.14: The Laplacian of u can be computed by differentiating under the integral sign in 1.13; Proposition 1.15 then shows that u is harmonic in B .

To prove that u is continuous on $\bar{B}$, fix $\eta \in S$ and $\varepsilon > 0$. Choose $\delta > 0$ such that $|f(\zeta) - f(\eta)| < \varepsilon$ whenever $|\zeta - \eta| < \delta$ (and $\zeta \in S$). For $x \in B$, (a) and (b) of Proposition 1.16 imply

$$\begin{aligned} |u(x) - u(\eta)| &= \left| \int_S P(x, \zeta) (f(\zeta) - f(\eta)) d\sigma(\zeta) \right| \\ &\leq \int_{|\zeta - \eta| \leq \delta} P(x, \zeta) |f(\zeta) - f(\eta)| d\sigma(\zeta) \\ &\quad + \int_{|\zeta - \eta| > \delta} P(x, \zeta) |f(\zeta) - f(\eta)| d\sigma(\zeta) \\ &\leq \varepsilon + 2\|f\|_\infty \int_{|\zeta - \eta| > \delta} P(x, \zeta) d\sigma(\zeta), \end{aligned}$$

where $\|f\|_\infty$ denotes the supremum of $|f|$ on S . The last term is less than ε for x sufficiently close to η (by Proposition 1.16(c)), proving that u is continuous at η . $\square$

We now prove a result stronger than that expressed in 1.11.

1.17 Theorem: *If u is a continuous function on $\bar{B}$ that is harmonic on B , then $u = P[u|_S]$ on B .*

PROOF: By 1.14, $u - P[u|_S]$ is harmonic on B and extends continuously to be 0 on S . The maximum principle (Corollary 1.5) now implies that $u - P[u|_S]$ is 0 on B . $\square$

Because translations and dilations preserve harmonic functions, we can assert the following for every ball $B(a, r)$: Given a

continuous f on $\partial B(a, r)$, there exists a unique continuous function u on $\bar{B}(a, r)$, with u harmonic on $B(a, r)$, such that $u = f$ on $\partial B(a, r)$. In this case we say that u solves the Dirichlet problem on $\bar{B}(a, r)$ with boundary data f .

We now show that every harmonic function is infinitely differentiable. In dealing with differentiation in several variables the following notation is useful: A *multi-index* α is an n -tuple of non-negative integers $(\alpha_1, \dots, \alpha_n)$. The partial differentiation operator D^α is defined to be $D_1^{\alpha_1} \dots D_n^{\alpha_n}$ (D_j^0 denotes the identity operator). For each $\zeta \in S$, the function $P(\cdot, \zeta)$ is infinitely differentiable on B ; we denote its α^{th} partial derivative by $D^\alpha P(\cdot, \zeta)$ (here ζ is held fixed).

If u is continuous on $\bar{B}$ and harmonic on B , then

$$u(x) = \int_S P(x, \zeta) u(\zeta) d\sigma(\zeta)$$

for every $x \in B$. Differentiating under the integral, we easily see that $u \in C^\infty(B)$; the formula

$$1.18 \quad D^\alpha u(x) = \int_S D^\alpha P(x, \zeta) u(\zeta) d\sigma(\zeta)$$

holds for every $x \in B$ and every multi-index α .

The preceding argument applies to any ball after a translation and dilation. As a consequence, every harmonic function is infinitely differentiable.

The following theorem should remind the reader of the behavior of a uniformly convergent sequence of holomorphic functions.

1.19 Theorem: Suppose (u_m) is a sequence of harmonic functions on Ω such that u_m converges uniformly to a function u on each compact subset of Ω . Then u is harmonic on Ω . Moreover, for every multi-index α , $D^\alpha u_m$ converges uniformly to $D^\alpha u$ on each compact subset of Ω .

PROOF: Given $\bar{B}(a, r) \subset \Omega$, we need only show that u is harmonic on $B(a, r)$ and that for every multi-index α , $D^\alpha u_m$ converges uniformly to $D^\alpha u$ on each compact subset of $B(a, r)$. Without loss of generality, we assume $B(a, r) = B$.

We then know that

$$u_m(x) = \int_S P(x, \zeta) u_m(\zeta) d\sigma(\zeta)$$

for every $x \in B$ and every m . Taking the limit of both sides, we obtain

$$u(x) = \int_S P(x, \zeta) u(\zeta) d\sigma(\zeta)$$

for every $x \in B$. Thus u is harmonic on B .

Let α be a multi-index and let $x \in B$. Then

$$\begin{aligned} D^\alpha u_m(x) &= \int_S D^\alpha P(x, \zeta) u_m(\zeta) d\sigma(\zeta) \\ &\rightarrow \int_S D^\alpha P(x, \zeta) u(\zeta) d\sigma(\zeta) = D^\alpha u(x). \end{aligned}$$

If K is a compact subset of B , then $D^\alpha P$ is uniformly bounded on $K \times S$, and so the convergence of $D^\alpha u_m$ to $D^\alpha u$ is uniform on K , as desired. $\square$

Converse of the Mean-Value Property

We have seen that every harmonic function has the mean-value property. In this section, we use the solvability of the Dirichlet problem for the ball to prove that harmonic functions are the only continuous functions having the mean-value property. In fact, the following theorem shows that a continuous function satisfying a weak form of the mean-value property must be harmonic.

1.20 Theorem: *Suppose u is a continuous function on Ω . If for each $x \in \Omega$ there is a sequence of positive numbers $r_j \rightarrow 0$ such that*

$$u(x) = \int_S u(x + r_j \zeta) d\sigma(\zeta)$$

for all j , then u is harmonic on Ω .

PROOF: Without loss of generality, we can assume that u is real valued. Suppose that $\bar{B}(a, R) \subset \Omega$. Let v solve the Dirichlet problem on $\bar{B}(a, R)$ with boundary data u on $\partial B(a, R)$. We will complete the proof by showing that $v = u$ on $B(a, R)$.

Suppose that $v - u$ is positive at some point of $\bar{B}(a, R)$. Let E be the subset of $\bar{B}(a, R)$ where $v - u$ attains its maximum. Because E is compact, E contains a point x farthest from a . Clearly $x \in B(a, R)$, so there exists a ball $B(x, r) \subset B(a, R)$ such that $u(x)$ equals the average of u over $\partial B(x, r)$.

Because v is harmonic, we have

$$(v - u)(x) = \int_S (v - u)(x + r\zeta) d\sigma(\zeta).$$

But $(v - u)(x + r\zeta) \leq (v - u)(x)$ for all $\zeta \in S$, with strict inequality on a non-empty open subset of S (because of how x was chosen), contradicting the equation above. Thus $v - u \leq 0$ on $\bar{B}(a, R)$. Similarly, $v - u \geq 0$ on $\bar{B}(a, R)$. $\square$

A similar proof shows that if u is continuous on Ω and satisfies a local mean-value property with respect to volume measure, then u is harmonic on Ω ; see Exercise 17 of this chapter.

The hypothesis of continuity is needed in Theorem 1.20, as the following example shows: Let $\Omega = \mathbf{R}^n$ and define u by

$$u(x) = \begin{cases} 1 & \text{if } x_n > 0 \\ 0 & \text{if } x_n = 0 \\ -1 & \text{if } x_n < 0. \end{cases}$$

Then $u(x)$ equals the average of u over every sphere centered at x if $x_n = 0$, and $u(x)$ equals the average of u over all sufficiently small spheres centered at x if $x_n \neq 0$. But u is not even continuous, much less harmonic, on $\mathbf{R}^n$.

In the following theorem we replace the continuity assumption with the weaker condition of local integrability (a function is *locally integrable* on Ω if it is Lebesgue integrable on every compact subset of Ω). However, we now require that the averaging property (with respect to volume measure) hold for every radius.

1.21 Theorem: *If u is a locally integrable function on Ω such that*

$$u(a) = \frac{1}{V(B(a, r))} \int_{B(a, r)} u dV$$

whenever $\bar{B}(a, r) \subset \Omega$, then u is harmonic on Ω .

PROOF: By Exercise 17 of this chapter, we need only show that u is continuous on Ω . Fix $a \in \Omega$ and let (a_j) be a sequence in Ω converging to a . Let K be a compact subset of Ω with a in the interior of K . Then there exists an $r > 0$ such that $B(a_j, r) \subset K$ for all sufficiently large j . Because u is integrable on K , the dominated convergence theorem shows that

$$\begin{aligned} u(a_j) &= \frac{1}{V(B(a, r))} \int_{B(a_j, r)} u \, dV \\ &= \frac{1}{V(B(a, r))} \int_K u \chi_{B(a_j, r)} \, dV \\ &\rightarrow \frac{1}{V(B(a, r))} \int_K u \chi_{B(a, r)} \, dV = u(a) \end{aligned}$$

(as usual, χ_E denotes the function that is 1 on E and 0 off E). Thus u is continuous on Ω , as desired. $\square$

Real Analyticity and Homogeneous Expansions

We saw in the last section that harmonic functions are infinitely differentiable. A much stronger property will be established in this section: Harmonic functions are real analytic. Roughly speaking, a function is real analytic if it is locally expressible as a power series in the coordinate variables $x_1, x_2, \dots, x_n$ of $\mathbf{R}^n$.

To make this more precise, we need to discuss what is meant by a series of complex numbers of the form $\sum c_\alpha$, where the summation is over all multi-indices α . (The full range of multi-indices will always be intended in a series unless indicated otherwise.) The problem is that there is no natural ordering of the set of all multi-indices when $n > 1$. However, suppose we know that $\sum c_\alpha$ is absolutely convergent, i.e., that

$$\sup \sum_{\alpha \in F} |c_\alpha| < \infty,$$

where the supremum is taken over all finite subsets F of multi-indices. All orderings of multi-indices $\alpha(1), \alpha(2), \dots$ then yield the same value for $\sum_{j=1}^{\infty} c_{\alpha(j)}$, hence we may unambiguously write $\sum c_\alpha$ for this value. We will only be concerned with such absolutely con-

vergent series.

The following notation will be convenient when dealing with multiple power series: For $x \in \mathbf{R}^n$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ a multi-index, define

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n},$$

$$\alpha! = \alpha_1! \alpha_2! \dots \alpha_n!,$$

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n.$$

A function f on Ω is *real analytic* on Ω if for every $a \in \Omega$ there exist complex numbers c_α such that

$$f(x) = \sum c_\alpha (x - a)^\alpha$$

for all x in a neighborhood of a , the series converging absolutely in this neighborhood.

Some basic properties of such series are contained in the next proposition. Here it will be convenient to center the power series at $a = 0$, and to define

$$R(y) = \{x \in \mathbf{R}^n : |x_j| < |y_j|, \quad j = 1, 2, \dots, n\}$$

for $y \in \mathbf{R}^n$; $R(y)$ is the n -dimensional open rectangle centered at 0 with “corner y ”. To avoid trivialities we will assume that each component of y is nonzero.

1.22 Theorem: Suppose $\{c_\alpha y^\alpha\}$ is a bounded set. Then:

(a) For every multi-index β , the series

$$\sum_{\alpha} D^{\beta} (c_{\alpha} x^{\alpha})$$

converges absolutely on $R(y)$ and uniformly on compact subsets of $R(y)$.

(b) The function f defined by $f(x) = \sum c_\alpha x^\alpha$ for $x \in R(y)$ is infinitely differentiable on $R(y)$. Moreover,

$$D^{\beta} f(x) = \sum_{\alpha} D^{\beta} (c_{\alpha} x^{\alpha})$$

for all $x \in R(y)$ and for every multi-index β . Furthermore, $c_\alpha = D^\alpha f(0)/\alpha!$ for every multi-index α .

REMARKS: 1. To say the preceding series converges uniformly on a set means that every ordering of the series converges uniformly on this set in the usual sense.

2. The theorem shows that every derivative of a real-analytic function is real analytic, and that if $\sum a_\alpha x^\alpha = \sum b_\alpha x^\alpha$ for all x in a neighborhood of 0, then $a_\alpha = b_\alpha$ for all α .

PROOF OF THEOREM 1.22: We first observe that on the rectangle $R((1, 1, \dots, 1))$,

$$\sum_{\alpha} D^{\beta}(x^{\alpha}) = D^{\beta}[(1 - x_1)^{-1}(1 - x_2)^{-1} \dots (1 - x_n)^{-1}]$$

for every multi-index β , as the reader should verify. (Start with $\beta = 0$.)

Now assume that $|c_\alpha y^\alpha| \leq M$ for every α . If K is a compact subset of $R(y)$, then $K \subset R(ty)$ for some $t \in (0, 1)$. Thus for every $x \in K$ and every multi-index α ,

$$|c_\alpha x^\alpha| \leq t^{|\alpha|} |c_\alpha y^\alpha| \leq M t^{|\alpha|}.$$

By the preceding paragraph, $\sum t^{|\alpha|} = (1 - t)^{-n} < \infty$, establishing the absolute and uniform convergence of $\sum c_\alpha x^\alpha$ on K . Similar reasoning, with a little more bookkeeping, applies to $\sum D^{\beta}(c_\alpha x^\alpha)$. This completes the proof of (a).

Letting $f(x) = \sum c_\alpha x^\alpha$ for $x \in R(y)$, the uniform convergence on compact subsets of $R(y)$ of the series $\sum D^{\beta}(c_\alpha x^\alpha)$ for every β shows that $f \in C^\infty(R(y))$, and that $D^{\beta}f(x) = \sum D^{\beta}(c_\alpha x^\alpha)$ in $R(y)$ for every β . The formula for the Taylor coefficients c_α follows from this by computing the derivatives of f at 0. $\square$

A word of caution: Theorem 1.22 does not assert that rectangles are the natural domains of convergence of multiple power series. For example, in two dimensions the domain of convergence of $\sum_{j=1}^{\infty} (x_1 x_2)^j$ is $\{(x_1, x_2) \in \mathbf{R}^2 : |x_1 x_2| < 1\}$.

The next theorem shows that real-analytic functions enjoy certain properties not shared by all C^∞ -functions.

1.23 Theorem: Suppose Ω is connected, f is real analytic in Ω , and $f = 0$ on a non-empty open subset of Ω . Then $f \equiv 0$ in Ω .

PROOF: Let ω denote the interior of $\{x \in \Omega : f(x) = 0\}$. Then ω is an open subset of Ω . The vanishing of all derivatives of f on ω shows that ω is also closed in Ω : If $a \in \Omega$ is a limit point of ω , then all derivatives of f vanish at a by continuity, implying that the power series of f at a is identically zero; hence $a \in \omega$. Since ω is non-empty by hypothesis, we must have $\omega = \Omega$ by connectivity, giving $f \equiv 0$ in Ω . $\square$

1.24 Theorem: *If u is harmonic in Ω , then u is real analytic in Ω .*

PROOF: It suffices to show that if u is harmonic on $\bar{B}$, then u has a power series expansion converging to u in a neighborhood of 0.

The main idea here is exactly the same as in one complex variable: We use the Poisson integral representation of u and expand the Poisson kernel in a power series. Unfortunately the details are not as simple as in the case of the Cauchy integral formula.

Suppose that $|x| < \sqrt{2} - 1$ and $\zeta \in S$. Then $0 < |x - \zeta|^2 < 2$, and thus

$$P(x, \zeta) = (1 - |x|^2)(|x - \zeta|^2)^{-n/2} = (1 - |x|^2) \sum_{m=0}^{\infty} c_m(|x|^2 - 2x \cdot \zeta)^m,$$

where $\sum_{m=0}^{\infty} c_m(t - 1)^m$ is the Taylor series of $t^{-n/2}$ on the interval $(0, 2)$, expanded about the point 1. After expanding the terms $(|x|^2 - 2x \cdot \zeta)^m$ and rearranging (permissible, since we have all of the absolute convergence one could ask for), the Poisson kernel takes the form

$$P(x, \zeta) = \sum_{\alpha} x^{\alpha} q_{\alpha}(\zeta),$$

for $x \in (\sqrt{2} - 1)B$ and $\zeta \in S$, where each q_{α} is a polynomial. This latter series converges uniformly on S for each $x \in (\sqrt{2} - 1)B$.

Thus if u is harmonic on $\bar{B}$,

$$\begin{aligned} u(x) &= \int_S P(x, \zeta) u(\zeta) d\sigma(\zeta) \\ &= \sum_{\alpha} \left(\int_S u q_{\alpha} d\sigma \right) x^{\alpha} \end{aligned}$$

for all $x \in (\sqrt{2} - 1)B$. This is the desired expansion of u near 0. $\square$

Unfortunately, the multiple power series at 0 of a function harmonic in B need not converge in all of B . For example, the function $u(z) = 1/(1 - z)$ is holomorphic (hence harmonic) in the open unit disk of the complex plane. Writing $z = x + iy = (x, y) \in \mathbf{R}^2$, we have

$$u(z) = \sum_{m=0}^{\infty} (x + iy)^m = \sum_{m=0}^{\infty} \sum_{j=0}^m \binom{m}{j} x^j (iy)^{m-j}$$

for $z \in B_2$. As a multiple power series, the last sum converges absolutely if and only if $|x| + |y| < 1$, and hence does not converge in all of B_2 . The reader should perhaps take a moment to meditate on the difference between the “real-analytic” and “holomorphic” power series of u .

As mentioned earlier, the real analyticity of harmonic functions allows us to prove a local maximum principle.

1.25 The Local Maximum Principle: *Suppose Ω is connected, u is real valued and harmonic in Ω , and u has a local maximum in Ω . Then u is constant in Ω .*

PROOF: If u has a local maximum at $a \in \Omega$, then there exists a ball $B(a, r) \subset \Omega$ such that $u \leq u(a)$ in $B(a, r)$. By Theorem 1.4, u is constant in $B(a, r)$. Because u is real analytic in Ω , $u \equiv u(a)$ in Ω by Theorem 1.23. $\square$

Knowing that harmonic functions locally have power series expansions enables us to express them locally as sums of homogeneous harmonic polynomials. This has many interesting consequences, as we will see later. In the remainder of this section we develop a few basic results, starting with a brief discussion of homogeneous polynomials.

A polynomial is by definition a finite linear combination of monomials x^α . A polynomial p of the form

$$p(x) = \sum_{|\alpha|=m} c_\alpha x^\alpha$$

is said to be *homogeneous of degree m* ; here we allow m to be any nonnegative integer. Equivalently, a polynomial p is homogeneous of degree m if $p(tx) = t^m p(x)$ for all $t \in \mathbf{R}$ and all $x \in \mathbf{R}^n$. This last formulation shows that a homogeneous polynomial is determined by

its restriction to S : If p and q are homogeneous of degree m and $p = q$ on S , then $p = q$ on $\mathbf{R}^n$. (This is not true of polynomials in general; for example, $1 - |x|^2 \equiv 0$ on S .) Note also that if p is a homogeneous polynomial of degree m , then so is $p \circ T$ for every linear map T from $\mathbf{R}^n$ to $\mathbf{R}^n$.

It is often useful to express functions as infinite sums of homogeneous polynomials. Here is a simple uniqueness result for such sums.

1.26 Proposition: *Let $r > 0$. If p_m and q_m are homogeneous polynomials of degree m , $m = 0, 1, \dots$, and if*

$$\sum_{m=0}^{\infty} p_m(x) = \sum_{m=0}^{\infty} q_m(x)$$

for all $x \in rB$ (both series converging pointwise in rB), then $p_m \equiv q_m$ for every m .

PROOF: Fix $\zeta \in S$. Since the two series above converge and are equal at each point in rB , we have

$$\sum_{m=0}^{\infty} p_m(\zeta)t^m = \sum_{m=0}^{\infty} q_m(\zeta)t^m$$

for all $t \in (-r, r)$. By the uniqueness of coefficients of power series in one variable, $p_m(\zeta) = q_m(\zeta)$ for every m . This is true for every $\zeta \in S$, and thus $p_m = q_m$ on S for all m . By the preceding remarks, $p_m = q_m$ on $\mathbf{R}^n$ for every m . $\square$

Suppose now that u is harmonic near 0. Letting

$$p_m(x) = \sum_{|\alpha|=m} \frac{D_\alpha u(0)}{\alpha!} x^\alpha,$$

we see from Theorem 1.24 that

$$u(x) = \sum_{m=0}^{\infty} p_m(x)$$

for x near 0. Because each p_m is homogeneous of degree m , the latter series is called the *homogeneous expansion* of u at 0. Remarkably,

the harmonicity of u implies that each p_m is harmonic. To see this, observe that $\Delta u = \sum \Delta p_m \equiv 0$ near 0, and that each Δp_m is homogeneous of degree $m - 2$ for $m \geq 2$ (and is 0 for $m < 2$). From 1.26 we conclude $\Delta p_m \equiv 0$ for every m . We have thus represented u near 0 as an infinite sum of homogeneous harmonic polynomials.

Translating this local result from 0 to any other point in the domain of u , we have the following theorem.

1.27 Theorem: *Suppose u is harmonic in Ω and $a \in \Omega$. Then there exist harmonic homogeneous polynomials p_m of degree m such that*

$$1.28 \quad u(x) = \sum_{m=0}^{\infty} p_m(x - a)$$

for all x near a , the series converging absolutely and uniformly near a .

Homogeneous expansions are better behaved than are multiple power series. In fact, we shall see later that if u is harmonic in Ω and $B(a, r) \subset \Omega$, then the homogeneous expansion 1.28 is valid for all $x \in B(a, r)$. This is of course reminiscent of the standard power series result for holomorphic functions of one complex variable. Indeed, if u is holomorphic in $\Omega \subset \mathbf{R}^2 = \mathbf{C}$, then by the uniqueness of homogeneous expansions, 1.28 is precisely the holomorphic power series of u in $B(a, r)$.

Origin of the Term “Harmonic”

The word “harmonic” is commonly used to describe a quality of sound. Harmonic functions derive their name from a rather round-about connection they have with one source of sound—a vibrating string.

Physicists label the movement of a point on a vibrating string “harmonic motion”. Such motion may be described using sine and cosine functions, and in this context the sine and cosine functions are sometimes called *harmonics*. In classical Fourier analysis, functions on the unit circle are expanded in terms of sines and cosines. Analogous expansions exist on the sphere in $\mathbf{R}^n$, $n > 2$, in terms of homogeneous harmonic polynomials (see Chapter 5). Because these

polynomials play the same role on the sphere that the harmonics sine and cosine play on the circle, they are called *spherical harmonics*. The term “spherical harmonic” was apparently first used in this context by William Thomson (Lord Kelvin) and Peter Tait (see [12], Appendix B). By the early 1900s, the word “harmonic” was applied not only to homogeneous polynomials with zero Laplacian, but to any solution of Laplace’s equation.

Exercises

1. (a) Show that if u and v are real-valued harmonic functions, then uv is harmonic if and only if $\nabla u \cdot \nabla v \equiv 0$.
 (b) Suppose Ω is connected and u is a real-valued harmonic function on Ω such that u^2 is harmonic. Prove that u is constant. Is this still true without the hypothesis that u is real valued?
2. Show that $\Delta(|x|^p) = p(p+n-2)|x|^{p-2}$.
3. (a) *The Laplacian in polar coordinates:* Suppose u is a twice continuously differentiable function of two real variables. Let $U(r, \theta) = u(r \cos \theta, r \sin \theta)$. Show that

$$\Delta u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial U}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 U}{\partial \theta^2}.$$

- (b) *The Laplacian in spherical coordinates:* Suppose u is a twice continuously differentiable function of three real variables. Let $U(\rho, \theta, \varphi) = u(\rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi)$. Show that

$$\Delta u = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial U}{\partial \rho} \right) + \frac{1}{\rho^2 \sin \varphi} \frac{\partial}{\partial \varphi} \left(\sin \varphi \frac{\partial U}{\partial \varphi} \right) + \frac{1}{\rho^2 \sin^2 \varphi} \frac{\partial^2 U}{\partial \theta^2}.$$

4. Suppose g is a real-valued function in $C^2(\mathbf{R}^n)$ and $f \in C^2(\mathbf{R})$. Prove that

$$\Delta(f \circ g)(x) = f''(g(x))|\nabla g(x)|^2 + f'(g(x))\Delta g(x).$$

5. Let u be harmonic on $\mathbf{R}^2$. Show that if f is holomorphic or conjugate holomorphic on $\mathbf{C}$, then $u \circ f$ is harmonic.

6. Suppose u is real valued and harmonic on B_2 . For $(x, y) \in B_2$, define

$$v(x, y) = \int_0^y (D_1 u)(x, t) dt - \int_0^x (D_2 u)(t, 0) dt.$$

Show that $u + iv$ is holomorphic on B_2 .

7. Suppose that u is harmonic on Ω . Define a function v on Ω by $v(x) = x \cdot \nabla u(x)$. Prove that v is harmonic on Ω .
8. Let $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a linear transformation such that $u \circ T$ is harmonic on $\mathbf{R}^n$ whenever u is harmonic on $\mathbf{R}^n$. Prove that T is a scalar multiple of an orthogonal transformation.
9. Suppose that Ω is connected and that u is real valued and harmonic on Ω . Show that if u is nonconstant on Ω , then $u(\Omega)$ is open in $\mathbf{R}$. (Thus u is an open mapping from Ω to $\mathbf{R}$.)
10. Suppose Ω is bounded and $\partial\Omega$ is connected. Show that if u is a real-valued continuous function on $\bar{\Omega}$ that is harmonic on Ω , then $u(\Omega) \subset u(\partial\Omega)$. Is this true for complex-valued u ?
11. A function is called *radial* if its value at x depends only on $|x|$. Prove that a radial harmonic function on B is constant.
12. Give another proof that $\int_S P(x, \zeta) d\sigma(\zeta) = 1$ for every $x \in B$ by showing that the function $x \mapsto \int_S P(x, \zeta) d\sigma(\zeta)$ is harmonic and radial on B .
13. Show that $P[f \circ T] = P[f] \circ T$ for every $f \in C(S)$ and every orthogonal transformation T .
14. Find the Poisson kernel for the ball $B(a, R)$.
15. Assuming Proposition 1.15, show that $P[f]$ is harmonic on B by using the converse of the mean-value property (1.20).
16. Use the mean-value property and its converse to give another proof that the uniform limit of a sequence of harmonic functions is harmonic.

17. Suppose u is a continuous function on Ω , and that for each $x \in \Omega$ there is a sequence of positive numbers $r_j \rightarrow 0$ such that

$$u(x) = \frac{1}{V(B(x, r_j))} \int_{B(x, r_j)} u \, dV$$

for each j . Prove that u is harmonic on Ω .

18. (a) *One-Radius Theorem*: Suppose that u is continuous on $\bar{B}$ and that for every $x \in B$, there exists a positive number $r(x) \leq 1 - |x|$ such that

$$u(x) = \int_S u(x + r(x)\zeta) \, d\sigma(\zeta).$$

Prove that u is harmonic on B .

(b) Show that the one-radius theorem fails if the assumption “ u is continuous on $\bar{B}$ ” is relaxed to “ u is continuous on B ”. (*Hint suggested to us by Walter Rudin*: For a counterexample in the $n = 2$ case, set $u(x) = a_m + b_m \log |x|$ on the annulus $\{1 - 2^{-m} \leq |x| \leq 1 - 2^{-m-1}\}$, where the constants a_m, b_m are chosen inductively. Proceed analogously when $n > 2$.)

19. *Hopf Lemma*: Suppose that u is real valued, nonconstant, and harmonic on $\bar{B}$. Show that if u attains its maximum value on $\bar{B}$ at $\zeta \in S$, then there is a positive constant c such that

$$u(\zeta) - u(r\zeta) \geq c(1 - r)$$

for all $r \in (0, 1)$. Conclude that $(D_{\mathbf{n}}u)(\zeta) > 0$; here $\mathbf{n}$ is the outer unit normal to S at ζ .

20. Show that a polynomial p is homogeneous of degree m if and only if $\nabla p \cdot x = mp$.
21. Suppose that $\sum c_\alpha x^\alpha$ converges in $R(y)$. Prove that $\sum c_\alpha x^\alpha$ is real analytic in $R(y)$.
22. A function $u: \Omega \rightarrow \mathbf{R}^m$ is said to be real analytic if each component of u is real analytic. Prove that the composition of real-analytic functions is real analytic. (Thus sums, products, quotients, ... of real-analytic functions are real analytic.)

23. Let m be a positive integer. Characterize all real-analytic functions u on $\mathbf{R}^n$ such that $u(tx) = t^m u(x)$ for all $x \in \mathbf{R}^n$ and all $t \in \mathbf{R}$.
24. Show that the power series expansion of a function harmonic on $\mathbf{R}^n$ converges everywhere on $\mathbf{R}^n$.

CHAPTER 2

Bounded Harmonic Functions

Liouville's Theorem

Liouville's Theorem in complex analysis states that a bounded holomorphic function on $\mathbf{C}$ is constant. A similar result holds for harmonic functions on $\mathbf{R}^n$. The simple proof given below is taken from Edward Nelson's paper [7], which is one of the rare mathematics papers not containing a single mathematical symbol.

2.1 Liouville's Theorem: *A bounded harmonic function on $\mathbf{R}^n$ is constant.*

PROOF: Suppose u is a harmonic function on $\mathbf{R}^n$, bounded by M . Let $x \in \mathbf{R}^n$ and let $r > 0$. By the volume version of the mean-value property (Theorem 1.3),

$$\begin{aligned} |u(x) - u(0)| &= \frac{1}{V(B(0, r))} \left| \int_{B(x, r)} u \, dV - \int_{B(0, r)} u \, dV \right| \\ &\leq M \frac{V(\mathcal{D}_r)}{V(B(0, r))}, \end{aligned}$$

where $\mathcal{D}_r$ denotes the symmetric difference of $B(x, r)$ and $B(0, r)$, so that $\mathcal{D}_r = [B(x, r) \cup B(0, r)] \setminus [B(x, r) \cap B(0, r)]$. The last expression above tends to 0 as $r \rightarrow \infty$. Thus $u(x) = u(0)$, and so u is constant. $\square$

Liouville's Theorem leads to an easy proof of a uniqueness theorem for bounded harmonic functions on open half-spaces. The upper half-space $H = H_n$ is the open subset of $\mathbf{R}^n$ defined by

$$H = \{x \in \mathbf{R}^n : x_n > 0\}.$$

In this setting we often identify $\mathbf{R}^n$ with $\mathbf{R}^{n-1} \times \mathbf{R}$, writing a typical point $z \in \mathbf{R}^n$ as $z = (x, y)$, where $x \in \mathbf{R}^{n-1}$ and $y \in \mathbf{R}$. We also identify ∂H with $\mathbf{R}^{n-1}$.

The next result shows that a continuous function on $\bar{H}$ that is bounded and harmonic on H is determined by its boundary values.

2.2 Corollary: *Suppose u is a continuous bounded function on $\bar{H}$ that is harmonic on H . If $u = 0$ on ∂H , then $u \equiv 0$ on $\bar{H}$.*

PROOF: For $x \in \mathbf{R}^{n-1}$ and $y < 0$, define $u(x, y) = -u(x, -y)$, thereby extending u to a bounded continuous function defined on all of $\mathbf{R}^n$. Clearly u satisfies the local mean-value property specified in Theorem 1.20, so u is harmonic on $\mathbf{R}^n$. Liouville's Theorem (2.1) now shows that u is constant on $\mathbf{R}^n$. $\square$

The hypothesis of boundedness is needed in the preceding corollary; for example, the function $u(x, y) = y$ is continuous on $\bar{H}$, harmonic on H , and 0 on $\mathbf{R}^{n-1}$.

In Chapter 7 we will study harmonic functions on H in detail.

Isolated Singularities

Everyone knows that an isolated singularity of a bounded holomorphic function is removable. We now show that the same is true for bounded harmonic functions.

We call a point $a \in \Omega$ an *isolated singularity* of any function u defined on $\Omega \setminus \{a\}$. When u is harmonic on $\Omega \setminus \{a\}$, the isolated singularity a is said to be *removable* if u has a harmonic extension to Ω .

2.3 Theorem: *An isolated singularity of a bounded harmonic function is removable.*

PROOF: It suffices to show that if u is bounded and harmonic on $\bar{B} \setminus \{0\}$, then u has a harmonic extension to B . Without loss of

generality, we can assume that u is real valued. The only candidate for a harmonic extension of u to B is the Poisson integral $P[u|_S]$.

Assume first that $n > 2$. For $\varepsilon > 0$, define the harmonic function v_ε on $B \setminus \{0\}$ by

$$v_\varepsilon(x) = u(x) - P[u|_S](x) + \varepsilon(|x|^{2-n} - 1).$$

Observe that as $|x| \rightarrow 1$, we have $v_\varepsilon(x) \rightarrow 0$ (by 1.14), while the boundedness of u shows that $v_\varepsilon(x) \rightarrow +\infty$ as $x \rightarrow 0$. By Corollary 1.6 (with “lim sup” replaced by “lim inf”), $v_\varepsilon \geq 0$ in $B \setminus \{0\}$. Letting $\varepsilon \rightarrow 0$, we conclude that $u - P[u|_S] \geq 0$ in $B \setminus \{0\}$. Replacing u by $-u$, we also have $u - P[u|_S] \leq 0$, giving $u = P[u|_S]$ in $B \setminus \{0\}$. Thus $P[u|_S]$ is the desired harmonic extension of u to B .

The proof when $n = 2$ is the same, except that $(|x|^{2-n} - 1)$ should be replaced by $\log 1/|x|$. $\square$

Cauchy's Estimates

If f is holomorphic and bounded by M on a disk $B(a, r) \subset \mathbf{C}$, then

$$|f^{(m)}(a)| \leq \frac{m! M}{r^m}$$

for every nonnegative integer m ; these are Cauchy's Estimates from complex analysis. The next theorem gives the comparable results for harmonic functions defined on balls in $\mathbf{R}^n$.

2.4 Cauchy's Estimates: *Let α be a multi-index. Then there is a positive constant C_α such that*

$$|D^\alpha u(a)| \leq \frac{C_\alpha M}{r^{|\alpha|}}$$

for all functions u harmonic and bounded by M on $B(a, r)$.

PROOF: We can assume that $a = 0$. If u is harmonic and bounded by M on $\bar{B}$, then by 1.18 we have

$$\begin{aligned}
|D^\alpha u(0)| &= \left| \int_S D^\alpha P(0, \zeta) u(\zeta) d\sigma(\zeta) \right| \\
&\leq M \int_S |D^\alpha P(0, \zeta)| d\sigma(\zeta) \\
&= C_\alpha M,
\end{aligned}$$

where $C_\alpha = \int_S |D^\alpha P(0, \zeta)| d\sigma(\zeta)$.

If u is harmonic and bounded by M on $\bar{B}(0, r)$, then applying the result in the previous paragraph to the r -dilate u_r shows that

$$|D^\alpha u(0)| \leq \frac{C_\alpha M}{r^{|\alpha|}}.$$

Replacing r by $r - \varepsilon$ and letting ε decrease to 0, we obtain the same conclusion if u is harmonic on the open ball $B(0, r)$ and bounded by M there. $\square$

The following corollary shows that the derivatives of a bounded harmonic function on Ω cannot grow too fast near $\partial\Omega$. We let $d(a, E)$ denote the distance from a point a to a set E .

2.5 Corollary: *Let u be a bounded harmonic function on Ω , and let α be a multi-index. Then there is a constant C such that*

$$|D^\alpha u(a)| \leq \frac{C}{d(a, \partial\Omega)^{|\alpha|}}$$

for all $a \in \Omega$.

PROOF: For each $a \in \Omega$, apply Cauchy's Estimates (Theorem 2.4) with $r = d(a, \partial\Omega)$. $\square$

Normal Families

In complex analysis the term *normal family* refers to a collection of holomorphic functions with the property that every sequence in the collection contains a subsequence converging uniformly on compact subsets of the domain. The most useful result in this area (and the key tool in most proofs of the Riemann Mapping Theorem) states that a collection of holomorphic functions that is uniformly bounded on each compact subset of the domain is a normal family. We now prove the analogous result for harmonic functions.

2.6 Theorem: *If (u_m) is a sequence of harmonic functions on Ω that is uniformly bounded on each compact subset of Ω , then some subsequence of (u_m) converges uniformly on each compact subset of Ω .*

PROOF: The key to the proof is the following observation: There exists a constant $C < \infty$ such that for all u harmonic and bounded by M on any ball $B(a, 2r)$,

$$|u(x) - u(a)| \leq \left(\sup_{B(a,r)} |\nabla u| \right) |x - a| \leq \frac{CM}{r} |x - a|$$

for all $x \in B(a, r)$. The first inequality is standard from advanced calculus; the second inequality is a consequence of Cauchy's Estimates (2.4).

Now suppose $K \subset \Omega$ is compact, and let $r = d(K, \partial\Omega)/3$. Because the set $K_{2r} = \{x \in \mathbf{R}^n : d(x, K) \leq 2r\}$ is a compact subset of Ω , the sequence (u_m) is uniformly bounded by some $M < \infty$ on K_{2r} . Let $a, x \in K$ and assume $|x - a| < r$. Then $x \in B(a, r)$ and $|u_m| \leq M$ on $B(a, 2r) \subset K_{2r}$ for all m , and so we conclude from the first paragraph that

$$|u_m(x) - u_m(a)| \leq \frac{CM}{r} |x - a|$$

for all m . It follows that the sequence (u_m) is equicontinuous on K .

To finish, choose compact sets $K_1 \subset K_2 \subset \cdots \subset \Omega$ whose interiors cover Ω . Because (u_m) is equicontinuous on K_1 , the Arzela-Ascoli Theorem ([9], Theorem 11.28) implies (u_m) contains a subsequence that converges uniformly on K_1 . Applying Arzela-Ascoli again, there is a subsequence of this subsequence converging uniformly on K_2 , and so on. If we list these subsequences one after another in rows, then the subsequence obtained by traveling down the diagonal converges uniformly on each K_j , and hence on each compact subset of Ω . $\square$

Note that by Theorem 1.19, the convergent subsequence obtained above converges to a harmonic function; furthermore, every partial derivative of this subsequence converges uniformly on each compact subset of Ω .

Theorem 2.6 is often useful in showing that certain extrema exist. For example, if $a \in \Omega$, then there exists a harmonic function v on Ω such that $|v| < 1$ on Ω and

$$|\nabla v(a)| = \sup\{|\nabla u(a)| : u \text{ is harmonic on } \Omega \text{ and } |u| < 1 \text{ on } \Omega\}.$$

Maximum Principles

Corollary 1.6 is the maximum principle in its most general form. It states that if u is a real-valued harmonic function on Ω and $u \leq M$ at the “boundary” of Ω , then $u \leq M$ on Ω . The catch is that we need to consider ∞ as a boundary point. (Again, the function $u(x, y) = y$ on H shows why this is necessary.) Often it is possible to ignore the point at infinity when u is bounded; the next result shows that this is always possible in two dimensions.

2.7 Theorem: *Suppose $\Omega \subset \mathbf{R}^2$ and $\Omega \neq \mathbf{R}^2$. If u is a real-valued, bounded harmonic function on Ω satisfying*

$$\mathbf{2.8} \quad \limsup_{k \rightarrow \infty} u(a_k) \leq M$$

for every sequence (a_k) in Ω converging to a point in $\partial\Omega$, then $u \leq M$ on Ω .

PROOF: Because $\Omega \neq \mathbf{R}^2$, $\partial\Omega$ is not empty. Let $\varepsilon > 0$, and choose a sequence in Ω converging to a point in $\partial\Omega$. By hypothesis, u is less than $M + \varepsilon$ on the tail end of this sequence. It follows that there is a closed ball $\bar{B}(a, r) \subset \Omega$ on which $u < M + \varepsilon$.

Define $\Omega' = \Omega \setminus \bar{B}(a, r)$, and set

$$v(z) = \log \left| \frac{z - a}{r} \right|$$

for $z \in \Omega'$. Then v is positive and harmonic on Ω' , and $v(z) \rightarrow \infty$ as $z \rightarrow \infty$ within Ω' .

For $t > 0$, we now define the harmonic function w_t on Ω' by

$$w_t = u - M - \varepsilon - tv.$$

By 2.8 and the preceding remarks, $\limsup w_t(a_k) < 0$ for every sequence (a_k) in Ω' converging to a point in $\partial\Omega'$, while the boundedness of u in Ω' shows that $w_t(a_k) \rightarrow -\infty$ for every sequence (a_k) converging to ∞ within Ω' . By Corollary 1.6, $w_t < 0$ on Ω' .

We now let $t \rightarrow 0$ to obtain $u \leq M + \varepsilon$ on Ω' . Because $u < M + \varepsilon$ on $\bar{B}(a, r)$, we have $u \leq M + \varepsilon$ on all of Ω . Finally, since ε is arbitrary, $u \leq M$ on Ω , as desired. $\square$

The higher-dimensional analogue of Theorem 2.7 fails. For an example, let $\Omega = \{x \in \mathbf{R}^n : |x| > 1\}$ and set $u(x) = 1 - |x|^{2-n}$. If $n > 2$, then u is a bounded harmonic function on $\bar{\Omega}$ that vanishes on $\partial\Omega$ but is not identically 0 on Ω . (In fact, u is never zero in Ω .)

The proof of Theorem 2.7 carries over to higher dimensions except for one key point: When $n = 2$, there exists a positive harmonic function v on $\mathbf{R}^n \setminus \bar{B}$ such that $v(z) \rightarrow \infty$ as $z \rightarrow \infty$. When $n > 2$, there exists no such v ; in fact, every positive harmonic function on $\mathbf{R}^n \setminus \bar{B}$ has a finite limit at ∞ when $n > 2$ (Theorem 4.7).

The following maximum principle is nevertheless valid in all dimensions. Recall that H_n denotes the upper half-space of $\mathbf{R}^n$.

2.9 Theorem: *Suppose $\Omega \subset H_n$. If u is a real-valued, bounded harmonic function on Ω satisfying*

$$\limsup_{k \rightarrow \infty} u(a_k) \leq M$$

for every sequence (a_k) in Ω converging to a point in $\partial\Omega$, then $u \leq M$ on Ω .

PROOF: For $(x, y) \in \Omega$, define

$$v(x, y) = \sum_{k=1}^{n-1} \log(x_k^2 + (y+1)^2).$$

Then v is positive and harmonic on Ω , and $v(z) \rightarrow \infty$ as $z \rightarrow \infty$ within H_n . Having obtained v , we can use the ideas in the proof of Theorem 2.7 to finish the proof. The details are even easier here and we leave them to the reader. $\square$

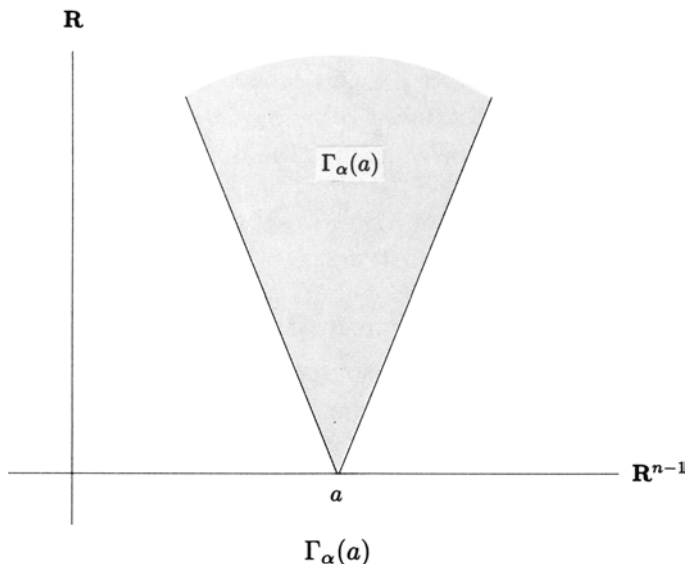
Limits Along Rays

We now apply some of the preceding results to study the boundary behavior of harmonic functions defined in the upper half-plane H_2 . We will need the notion of a nontangential limit, which for later purposes we define for functions on half-spaces of arbitrary dimension.

Given $a \in \mathbf{R}^{n-1}$ and $\alpha > 0$, set

$$\Gamma_\alpha(a) = \{(x, y) \in H_n : |x - a| < \alpha y\}.$$

Geometrically, $\Gamma_\alpha(a)$ is a cone with vertex a and axis of symmetry parallel to the y -axis.



We have $\Gamma_\alpha(a) \subset \Gamma_\beta(a)$ if $\alpha < \beta$, and H_n is the union of the sets $\Gamma_\alpha(a)$ as α ranges over $(0, \infty)$.

A function u defined on H_n is said to have a *nontangential limit* L at $a \in \mathbf{R}^{n-1}$ if, for every $\alpha > 0$, $u(z) \rightarrow L$ as $z \rightarrow a$ within $\Gamma_\alpha(a)$. The term “nontangential” is used because no curve in $\Gamma_\alpha(a)$ that approaches a can be tangent to ∂H_n . Exercise 16 of this chapter shows that a bounded harmonic function on H_n can have a nontangential limit at a point of ∂H_n even though the ordinary limit does not exist at that point.

The following theorem for bounded harmonic functions on H_2 asserts that a nontangential limit can be deduced from a limit along a certain one-dimensional set.

2.10 Theorem: *Suppose u is bounded and harmonic on H_2 . If $0 < \theta_1 < \theta_2 < \pi$ and*

$$\lim_{r \rightarrow 0} u(re^{i\theta_1}) = L = \lim_{r \rightarrow 0} u(re^{i\theta_2}),$$

then u has nontangential limit L at 0.

PROOF: We may assume $L = 0$.

If the theorem is false, then for some $\alpha > 0$, $u(z)$ fails to have limit 0 as $z \rightarrow 0$ within $\Gamma_\alpha(0)$. This means that there exists an $\varepsilon > 0$ and a sequence (z_j) tending to 0 within $\Gamma_\alpha(0)$ such that

$$|u(z_j)| > \varepsilon$$

for all j .

Define $K = [-\alpha, \alpha] \times \{1\}$, and write $z_j = r_j w_j$, where $w_j \in K$ and $r_j > 0$. Because $z_j \rightarrow 0$, we have $r_j \rightarrow 0$.

Setting $u_j(z) = u(r_j z)$, note that (u_j) is a uniformly bounded sequence of harmonic functions on H_2 . By Theorem 2.6, there exists a subsequence of (u_j) that converges uniformly on compact subsets of H_2 to a bounded harmonic function v on H_2 ; for simplicity we denote this subsequence by (u_j) as well.

Examining the limit function v , we see that

$$v(re^{i\theta_1}) = \lim_{j \rightarrow \infty} u_j(re^{i\theta_1}) = \lim_{j \rightarrow \infty} u(r_j re^{i\theta_1}) = 0$$

for all $r > 0$. Similarly, $v(re^{i\theta_2}) = 0$ for all $r > 0$. The reader may now be tempted to apply Theorem 2.7 to the region between the two rays; unfortunately we do not know that $v(z) \rightarrow 0$ as $z \rightarrow 0$ between the given rays. We avoid this problem by observing that the function $z \mapsto v(e^z)$ is bounded and harmonic on the strip $\Omega = \{z = x + iy : \theta_1 < y < \theta_2\}$, and that $v(e^z)$ extends continuously to $\bar{\Omega}$ with $v(e^z) = 0$ on $\partial\Omega$. By Theorem 2.7, $v(e^z) \equiv 0$ on Ω , and thus $v \equiv 0$ on H_2 .

The sequence (u_j) therefore converges to 0 uniformly on compact subsets of H_2 . In particular, $u_j \rightarrow 0$ uniformly on K . It follows that $u_j(w_j) = u(z_j) \rightarrow 0$ as $j \rightarrow \infty$, contradicting 2.11. $\square$

Does a limit along one ray suffice to give a nontangential limit in Theorem 2.10? To see that the answer is no, consider the bounded harmonic function u on H_2 defined by $u(re^{i\theta}) = \theta$ for $\theta \in (0, \pi)$. This function has a limit along each ray in H_2 emanating from the origin, but different rays yield different limits. (One ray will suffice for a bounded holomorphic function; see Exercise 21 of this chapter.)

Bounded Harmonic Functions on the Ball

In the last chapter we defined the Poisson integral $P[f]$ assuming that f is continuous on S . We can easily enlarge the class of functions f for which $P[f]$ is defined. For example, if f is a bounded (Borel) measurable function on S , then

$$P[f](x) = \int_S P(x, \zeta) f(\zeta) d\sigma(\zeta)$$

defines a bounded harmonic function on B ; we leave the verification to the reader.

Allowing bounded measurable boundary data gives us many more examples of bounded harmonic functions on B than could otherwise be obtained. For example, in Chapter 6 we will see that the extremal function in the Schwarz Lemma for harmonic functions is the Poisson integral of a bounded discontinuous function on S . In that chapter we will also prove a fundamental result: Given a bounded harmonic function u on B , there exists a bounded measurable f on S such that $u = P[f]$ on B .

Exercises

1. Give an example of a bounded harmonic function on B that is not uniformly continuous on B .
2. (a) Suppose that u is a harmonic function on $B \setminus \{0\}$ such that

$$|x|^{n-2}u(x) \rightarrow 0 \quad \text{as } x \rightarrow 0.$$

Prove that u has a removable singularity at 0.

(b) Suppose that u is a harmonic function on $B_2 \setminus \{0\}$ such that

$$u(x)/\log|x| \rightarrow 0 \quad \text{as } x \rightarrow 0.$$

Prove that u has a removable singularity at 0.

3. Suppose that u is harmonic on $\mathbf{R}^n$ and that $u(x, 0) = 0$ for all $x \in \mathbf{R}^{n-1}$. Prove that $u(x, -y) = -u(x, y)$ for all $(x, y) \in \mathbf{R}^n$.
4. Under what circumstances can a function harmonic on $\mathbf{R}^n$ vanish on the union of two hyperplanes?
5. Use Cauchy's Estimates (Theorem 2.4) to give another proof of Liouville's Theorem (Theorem 2.1).
6. Let K be a compact subset of Ω and let α be a multi-index. Show that there is a constant $C = C(\Omega, K, \alpha)$ such that

$$|D^\alpha u(a)| \leq C \sup\{|u(x)| : x \in \Omega\}$$

for every function u harmonic on Ω and every $a \in K$.

7. Suppose u is harmonic on $\mathbf{R}^n$ and $|u(x)| \leq A(1 + |x|^p)$ for all $x \in \mathbf{R}^n$, where A is a constant and $p \geq 0$. Prove that u is a polynomial of degree at most p .

8. Prove if (u_m) is a pointwise convergent sequence of harmonic functions on Ω that is uniformly bounded on each compact subset of Ω , then (u_m) converges uniformly on each compact subset of Ω .
9. Show that if u is the pointwise limit of a sequence of harmonic functions on Ω , then u is harmonic on a dense open subset of Ω . (Hint: Baire's Theorem.)
10. Let u be a bounded harmonic function on B . Prove that

$$\sup_{x \in B} (1 - |x|) |\nabla u(x)| < \infty.$$

11. The set of harmonic functions u on B satisfying the inequality in Exercise 10 is called the *harmonic Bloch space*. Prove the harmonic Bloch space is a Banach space under the norm defined by

$$\|u\| = \sup_{x \in B} (1 - |x|) |\nabla u(x)| + |u(0)|.$$

12. Give an example of an unbounded harmonic function in the harmonic Bloch space.
13. Prove that if u is in the harmonic Bloch space and α is a multi-index with $|\alpha| > 0$, then

$$\sup_{x \in B} (1 - |x|)^{|\alpha|} |D^\alpha u(x)| < \infty.$$

14. For $a \in B$, let B_a denote the ball with center a and radius $(1 - |a|)/2$. Prove that if u is harmonic on B , then u is in the harmonic Bloch space if and only if

$$\sup_{a \in B} \frac{1}{V(B_a)} \int_{B_a} |u - u(a)| dV < \infty.$$

15. Let Q denote the set of harmonic functions u on B such that $u(0) = 0$ and

$$\sup_{x \in B} (1 - |x|) |\nabla u(x)| \leq 1.$$

Prove that there exists a function $v \in Q$ such that

$$\int_S |v(\zeta/2)| d\sigma(\zeta) = \sup_{u \in Q} \int_S |u(\zeta/2)| d\sigma(\zeta).$$

16. Let $f(z) = e^{-i/z}$. Show that f is a bounded holomorphic function on H_2 , that f has a nontangential limit at the origin, but that f does not have a limit along some curve in H_2 terminating at the origin.
17. Suppose $0 < \theta_1 < \theta_2 < \pi$ and $L_1, L_2 \in \mathbb{C}$. Show that there is a bounded harmonic function u on H_2 such that $u(re^{i\theta_k}) \rightarrow L_k$ as $r \rightarrow 0$ for $k = 1, 2$.
18. Suppose u is a bounded harmonic function on H_2 , and that $\lim_{r \rightarrow 0} u(re^{i\theta_k}) = L_k$ for $k = 1, 2$, where $0 < \theta_1 < \theta_2 < \pi$. Show that $\lim_{r \rightarrow 0} u(re^{i\theta})$ exists for every $\theta \in (0, \pi)$, and evaluate this limit as a function of θ .
19. Define $f(z) = e^{i \log z}$, where $\log z$ denotes the principal-valued logarithm. Show that f is a bounded holomorphic function on H_2 whose real and imaginary parts fail to have a limit along every ray in H_2 emanating from the origin.
20. Let $\theta_0 \in (0, \pi)$. Prove that there exists a bounded harmonic function u on H_2 such that $\lim_{r \rightarrow 0} u(re^{i\theta})$ exists if and only if $\theta = \theta_0$. (Hint: Do this first for $\theta_0 = \pi/2$ by letting $u(z) = \operatorname{Re} e^{i \log z}$ and considering $u(x, y) - u(-x, y)$.)
21. Let f be a bounded holomorphic function on H_2 , and suppose $\lim_{r \rightarrow 0} f(re^{i\theta})$ exists for some $\theta \in (0, \pi)$. Prove that f has a nontangential limit at 0.
22. Let u be a bounded harmonic function on H_3 that has the same limit along two distinct rays in H_3 emanating from 0. Need u have a nontangential limit at 0?

23. Prove that when $n > 2$ there does not exist a harmonic function v on $\mathbf{R}^n \setminus \bar{B}$ such that $v(z) \rightarrow \infty$ as $z \rightarrow \infty$.
24. Let K denote a compact line segment contained in B_3 . Show that every bounded harmonic function on $B_3 \setminus K$ extends to be harmonic on B_3 .

CHAPTER 3

Positive Harmonic Functions

Liouville's Theorem

In Chapter 2 we proved that a bounded harmonic function on $\mathbf{R}^n$ is constant. We now improve that result.

3.1 Liouville's Theorem for Positive Harmonic Functions:
A positive harmonic function on $\mathbf{R}^n$ is constant.

PROOF OF 3.1: The proof is slightly more delicate than that given for bounded harmonic functions. Let u be a positive harmonic function defined on $\mathbf{R}^n$. Fix $x \in \mathbf{R}^n$. Let $r > |x|$, and let $\mathcal{D}_r$ denote the symmetric difference of the balls $B(x, r)$ and $B(0, r)$. The volume version of the mean-value property (1.3) shows that

$$u(x) - u(0) = \frac{1}{V(B(0, r))} \left[\int_{B(x, r)} u \, dV - \int_{B(0, r)} u \, dV \right].$$

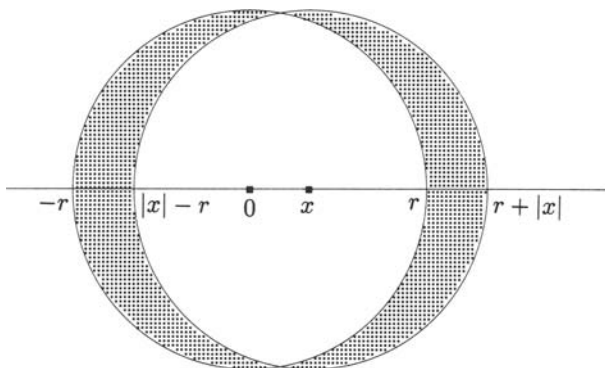
Because the integrals of u over $B(x, r) \cap B(0, r)$ cancel (see 3.2), we have

$$\begin{aligned} |u(x) - u(0)| &\leq \frac{1}{V(B(0, r))} \int_{\mathcal{D}_r} u \, dV \\ &\leq \frac{1}{V(B(0, r))} \int_{B(0, r+|x|) \setminus B(0, r-|x|)} u \, dV \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{V(B(0, r))} \left[\int_{B(0, r+|x|)} u \, dV - \int_{B(0, r-|x|)} u \, dV \right] \\
&= \frac{(r+|x|)^n - (r-|x|)^n}{r^n} u(0).
\end{aligned}$$

Note that the positivity of u was used in the first inequality.

Now letting $r \rightarrow \infty$, we see that $u(x) = u(0)$, proving that u is constant. $\square$



3.2 The symmetric difference $\mathcal{D}_r$ (shaded) of $B(x, r)$ and $B(0, r)$.

Liouville's Theorem for positive harmonic functions leads to an easy proof that a positive harmonic function on $\mathbf{R}^2 \setminus \{0\}$ is constant.

3.3 Corollary: A positive harmonic function on $\mathbf{R}^2 \setminus \{0\}$ is constant.

PROOF: If u is positive and harmonic on $\mathbf{R}^2 \setminus \{0\}$, then the function $z \mapsto u(e^z)$ is positive and harmonic on $\mathbf{R}^2 (= \mathbf{C})$ and hence (by 3.1) is constant. This proves that u is constant. $\square$

The higher-dimensional analogue of Corollary 3.3 fails; for example, the function $|x|^{2-n}$ is positive and harmonic on $\mathbf{R}^n \setminus \{0\}$ when $n > 2$. We will classify the positive harmonic functions on $\mathbf{R}^n \setminus \{0\}$ for $n > 2$ after the proof of Bôcher's Theorem; see Corollary 3.14.

Harnack's Inequality and Harnack's Principle

Positive harmonic functions cannot oscillate too much on a compact set $K \subset \Omega$ if Ω is connected; the precise statement is Harnack's Inequality (3.6). We first consider the important special case where Ω is the open unit ball.

3.4 Harnack's Inequality for the Ball: *If u is positive and harmonic on B , then*

$$\frac{1 - |x|}{(1 + |x|)^{n-1}} u(0) \leq u(x) \leq \frac{1 + |x|}{(1 - |x|)^{n-1}} u(0)$$

for all $x \in B$.

PROOF: If u is positive and harmonic on the closed unit ball $\bar{B}$, then

$$\begin{aligned} u(x) &= P[u|_S](x) \\ &= \int_S \frac{1 - |x|^2}{|x - \zeta|^n} u(\zeta) d\sigma(\zeta) \\ &\leq \frac{1 - |x|^2}{(1 - |x|)^n} \int_S u(\zeta) d\sigma(\zeta) \\ &= \frac{1 + |x|}{(1 - |x|)^{n-1}} u(0) \end{aligned}$$

for all $x \in B$. If u is positive and harmonic on B , apply the above to the dilates u_r and take the limit as $r \rightarrow 1$. This gives us the second inequality of the theorem. The first inequality is proved similarly. $\square$

For $0 \leq t < 1$, define $\alpha(t) = (1 - t)/(1 + t)^{n-1}$ and $\beta(t) = (1 + t)/(1 - t)^{n-1}$. After a translation and a dilation, 3.4 tells us that if u is positive and harmonic on $B(a, R)$, and $|x - a| \leq r < R$, then

3.5
$$\alpha(r/R)u(a) \leq u(x) \leq \beta(r/R)u(a).$$

3.6 Harnack's Inequality: *Suppose that Ω is connected and that K is a compact subset of Ω . Then there is a constant $C \in (1, \infty)$ such that*

$$\frac{1}{C} \leq \frac{u(y)}{u(x)} \leq C$$

for all points x and y in K and all positive harmonic functions u on Ω .

PROOF: We will prove that there is a constant $C \in (1, \infty)$ such that $u(y)/u(x) \leq C$ for all $x, y \in K$ and all positive harmonic functions u on Ω . Because x and y play symmetric roles, the other inequality will also hold.

For $(x, y) \in \Omega \times \Omega$, define

$$s(x, y) = \sup\{u(y)/u(x) : u \text{ is positive and harmonic on } \Omega\}.$$

We first show that $s < \infty$ on $\Omega \times \Omega$.

Fix $x \in \Omega$, and define

$$E = \{y \in \Omega : s(x, y) < \infty\}.$$

Because $x \in E$, E is not empty. If $y \in E$, we may choose $r > 0$ such that $B(y, 2r) \subset \Omega$. By 3.5, $u \leq \beta(1/2)u(y)$ on $B(y, r)$ for all positive harmonic u on Ω . We then have $B(y, r) \subset E$, proving that E is open. If $z \in \Omega$ is a limit point of E , there exists an $r > 0$ and a $y \in E$ such that $z \in B(y, r) \subset B(y, 2r) \subset \Omega$. By 3.5, $u(z) \leq \beta(1/2)u(y)$ for all positive harmonic u on Ω . We then have $z \in E$, proving that E is closed. The connectivity of Ω therefore shows that $E = \Omega$.

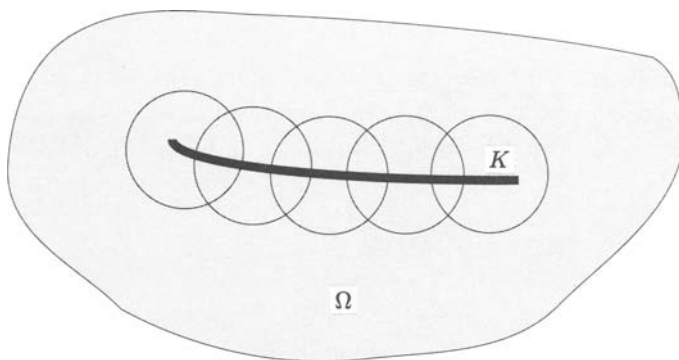
We now know that s is finite at every point of $\Omega \times \Omega$. Let $K \subset \Omega$ be compact, and let $(a, b) \in K \times K$. Then by 3.5,

$$\frac{u(y)}{u(x)} \leq \frac{\beta(1/2)u(b)}{\alpha(1/2)u(a)} \leq \frac{\beta(1/2)}{\alpha(1/2)}s(a, b)$$

for all (x, y) in a neighborhood of (a, b) , and for all positive harmonic functions u on Ω . Because $K \times K$ is covered by finitely many such neighborhoods, s is bounded above on $K \times K$, as desired. $\square$

Note that the constant C in 3.6 may depend upon Ω and K , but that C is independent of x , y , and u .

An intuitive way to remember Harnack's Inequality is shown in 3.7. Here we have covered K with a finite chain of overlapping balls (possible, since Ω is connected); to compare the values of a positive harmonic function at any two points in K , we can think of a finite chain of inequalities of the kind expressed in 3.5.



3.7 K covered by overlapping balls.

Harnack's Inequality leads to an important convergence theorem for harmonic functions known as Harnack's Principle. Consider for a moment a monotone sequence of continuous functions on Ω . The pointwise limit of such a sequence need not behave well—it could be infinite at some points and finite at other points. Even if it is finite everywhere, there is no reason to expect that our sequence converges uniformly on every compact subset of Ω . Harnack's Principle shows that none of this bad behavior can occur for a monotone sequence of harmonic functions.

3.8 Harnack's Principle: Suppose Ω is connected and (u_m) is a pointwise increasing sequence of harmonic functions on Ω . There are only two possibilities: Either $u_m(x) \rightarrow \infty$ for every $x \in \Omega$, or (u_m) converges uniformly on compact subsets of Ω to a function harmonic on Ω .

PROOF: Replacing u_m by $u_m - u_1 + 1$, we can assume that each u_m is positive on Ω . Set $u(x) = \lim_{m \rightarrow \infty} u_m(x)$ for each $x \in \Omega$.

Suppose $u(x) = \infty$ for some $x \in \Omega$. Let $y \in \Omega$. Then Harnack's Inequality (3.6), applied to the compact set $K = \{x, y\}$, shows

that there is a constant $C \in (1, \infty)$ such that $u_m(x) \leq Cu_m(y)$ for every m . Because $u_m(x) \rightarrow \infty$, we also have $u_m(y) \rightarrow \infty$, and so $u(y) = \infty$. This implies u is identically ∞ on Ω .

Now suppose u is finite everywhere on Ω . Let K be a compact subset of Ω . Fix $x \in K$. Harnack's Inequality (3.6) shows there is a constant $C \in (1, \infty)$ such that

$$u_m(y) - u_k(y) \leq C(u_m(x) - u_k(x))$$

for all $y \in K$, whenever $m > k$. This implies (u_m) is uniformly Cauchy on K , and thus $u_m \rightarrow u$ uniformly on K , as desired. Theorem 1.19 shows that the limit function u is harmonic on Ω . $\square$

Isolated Singularities

In this section we prove Bôcher's Theorem, which characterizes the behavior of positive harmonic functions in the neighborhood of an isolated singularity. Recall that when $n = 2$, the function $\log(1/|x|)$ is positive and harmonic in $B \setminus \{0\}$, while when $n > 2$, the function $|x|^{2-n}$ is positive and harmonic in $B \setminus \{0\}$. Roughly speaking, Bôcher's Theorem says that near an isolated singularity, a positive harmonic function must behave like one of these functions.

3.9 Bôcher's Theorem: *Suppose u is positive and harmonic in $B \setminus \{0\}$. Then there exists a function v harmonic on B and a constant $b \geq 0$ such that on $B \setminus \{0\}$,*

- (a) $u(x) = b \log(1/|x|) + v(x)$ (if $n = 2$);
- (b) $u(x) = b|x|^{2-n} + v(x)$ (if $n > 2$).

The next three lemmas will be used in the proof of Bôcher's Theorem. The first lemma describes the spherical averages of a function harmonic in a punctured ball. Given a continuous function u defined in $B \setminus \{0\}$, we define $A[u](x)$, the average of u over the sphere of radius $|x|$, by

$$A[u](x) = \int_S u(|x|\zeta) d\sigma(\zeta)$$

for $x \in B \setminus \{0\}$.

3.10 Lemma: *If u is harmonic in $B \setminus \{0\}$, then there are constants b and c such that on $B \setminus \{0\}$,*

- (a) $A[u](x) = b \log(1/|x|) + c$ (if $n = 2$);
 (b) $A[u](x) = b|x|^{2-n} + c$ (if $n > 2$).

In particular, $A[u]$ is harmonic in $B \setminus \{0\}$.

PROOF: Let ds denote surface-area measure (unnormalized). Define f on $(0, 1)$ by

$$f(r) = \int_S u(r\zeta) ds(\zeta);$$

so $A[u](x)$ is a constant multiple of $f(|x|)$. Because u is continuously differentiable on $B \setminus \{0\}$, we can compute f' by differentiating under the integral sign, obtaining

$$f'(r) = \int_S \zeta \cdot (\nabla u)(r\zeta) ds(\zeta) = r^{-n} \int_{rS} \tau \cdot (\nabla u)(\tau) ds(\tau).$$

Suppose $0 < r_0 < r_1 < 1$ and $\Omega = \{x \in \mathbf{R}^n : r_0 < |x| < r_1\}$. The divergence theorem, applied to ∇u , shows that

$$\int_{\partial\Omega} \mathbf{n} \cdot \nabla u ds = \int_{\Omega} \Delta u dV;$$

here $\mathbf{n}$ denotes the outward unit normal on Ω . Because u is harmonic on Ω , the right side of this equation is 0. Note also that $\partial\Omega = r_0S \cup r_1S$ and that $\mathbf{n} = -\tau/r_0$ on r_0S and $\mathbf{n} = \tau/r_1$ on r_1S . Thus the last equation above implies that

$$\frac{1}{r_0} \int_{r_0S} \tau \cdot (\nabla u)(\tau) ds(\tau) = \frac{1}{r_1} \int_{r_1S} \tau \cdot (\nabla u)(\tau) ds(\tau),$$

which means $f'(r)$ is a constant multiple of r^{1-n} (for $0 < r < 1$). This proves $f(r)$ is of the desired form. $\square$

An immediate consequence of Lemma 3.10 is that every radial harmonic function on $B \setminus \{0\}$ is one of the functions specified in (a) or (b) (a function is *radial* if its value at x depends only on $|x|$). Proof: If u is radial, then $u = A[u]$. (For another proof, see Exercise 13 of this chapter.)

The next lemma is a version of Harnack's Inequality that allows x and y to range over a noncompact set provided $|x| = |y|$.

3.11 Lemma: *There exists a constant $c > 0$ such that for every positive harmonic function u in $B \setminus \{0\}$,*

$$cu(y) < u(x)$$

whenever $0 < |x| = |y| \leq 1/2$.

PROOF: Harnack's Inequality (3.6), with $\Omega = B \setminus \{0\}$ and $K = (1/2)S$, shows there is a constant $c > 0$ such that for all positive harmonic u in $B \setminus \{0\}$, we have $cu(y) < u(x)$ whenever $|x| = |y| = 1/2$. Applying this result to the dilates $u_r, 0 < r < 1$, gives the desired conclusion. $\square$

The following result characterizes the positive harmonic functions on $B \setminus \{0\}$ that are identically zero on S . This is really the heart of our proof of Bôcher's Theorem.

3.12 Lemma: *If u is positive and harmonic in $B \setminus \{0\}$ and $u(x) \rightarrow 0$ as $|x| \rightarrow 1$, then there is a positive constant b such that on $B \setminus \{0\}$,*

- (a) $u(x) = b \log(1/|x|)$ (if $n = 2$);
- (b) $u(x) = b(|x|^{2-n} - 1)$ (if $n > 2$).

PROOF: By Lemma 3.10, we need only show that $u = A[u]$ in $B \setminus \{0\}$. Suppose we could show that $u \geq A[u]$ in $B \setminus \{0\}$. Then if there were a point $x \in B \setminus \{0\}$ such that $u(x) > A[u](x)$, we would have

$$A[u](x) > A[A[u]](x) = A[u](x),$$

a contradiction. Thus we need only prove that $u \geq A[u]$ in $B \setminus \{0\}$, which we now do.

Let c be the constant of Lemma 3.11. By Lemma 3.10, $u - cA[u]$ is harmonic in $B \setminus \{0\}$. By Lemma 3.11, $u(x) - cA[u](x) > 0$ if $0 < |x| \leq 1/2$, and clearly $u(x) - cA[u](x) \rightarrow 0$ as $|x| \rightarrow 1$ by our hypothesis on u . The minimum principle for harmonic functions (1.6) thus shows that $u - cA[u] > 0$ in $B \setminus \{0\}$.

We wish to iterate this result. For this purpose, define

$$g(t) = c + t(1 - c)$$

for $t \in [0, 1]$. Suppose we know

$$3.13 \quad w = u - tA[u] > 0$$

in $B \setminus \{0\}$ for some $t \in [0, 1]$. Since $w(x) \rightarrow 0$ as $|x| \rightarrow 1$, the preceding argument may be applied to w , yielding

$$w - cA[w] = u - g(t)A[u] > 0$$

in $B \setminus \{0\}$. This process may be continued. Letting $g^{(m)}$ denote the m^{th} iterate of g , we see that 3.13 implies

$$u - g^{(m)}(t)A[u] > 0$$

in $B \setminus \{0\}$ for $m = 1, 2, \dots$. But $g^{(m)}(t) \rightarrow 1$ as $m \rightarrow \infty$, for every $t \in [0, 1]$, so that 3.13 holding for some $t \in [0, 1]$ implies $u - A[u] \geq 0$ in $B \setminus \{0\}$. Since 3.13 obviously holds when $t = 0$, we have $u - A[u] \geq 0$ in $B \setminus \{0\}$, as desired. $\square$

Now we are ready to prove Bôcher's Theorem (3.9).

PROOF OF BÔCHER'S THEOREM: We first assume that $n > 2$ and that u is positive and harmonic on $\bar{B} \setminus \{0\}$. Define a harmonic function w on $B \setminus \{0\}$ by

$$w(x) = u(x) - P[u|_S](x) + |x|^{2-n} - 1.$$

As $|x| \rightarrow 1$, we have $w(x) \rightarrow 0$ (by 1.14), and as $|x| \rightarrow 0$, we have $w(x) \rightarrow +\infty$ (because u is positive and $P[u|_S]$ is bounded on $B \setminus \{0\}$). By the minimum principle (1.6), we conclude that w is positive on $B \setminus \{0\}$.

Lemma 3.12, applied to w , shows that $u(x) = b|x|^{2-n} + v(x)$ in $B \setminus \{0\}$ for some v harmonic in B and some constant b . Now letting $x \rightarrow 0$, we see that the positivity of u implies that b is nonnegative; we have thus proved Bôcher's Theorem in the case where u is positive and harmonic on $\bar{B} \setminus \{0\}$.

For the general positive harmonic u in $B \setminus \{0\}$, we may apply the result above to the dilate $u(x/2)$, so that

$$u(x/2) = b|x|^{2-n} + v(x)$$

in $B \setminus \{0\}$ for some v harmonic in B and some constant $b \geq 0$. This implies

$$u(x) = b2^{2-n}|x|^{2-n} + v(2x)$$

in $(1/2)B \setminus \{0\}$, which shows that $u(x) - b2^{2-n}|x|^{2-n}$ extends harmonically to $(1/2)B$, and hence to B . Thus the proof of Bôcher's Theorem is complete in the case where $n > 2$. The proof of the $n = 2$ case is the same, except that $\log(1/|x|)$ should be replaced by $|x|^{2-n}$. $\square$

In Chapter 9 we will give another proof of Bôcher's Theorem (see 9.8).

We conclude this section by characterizing the positive harmonic functions on $\mathbf{R}^n \setminus \{0\}$ for $n > 2$. (Recall that by 3.3, a positive harmonic function on $\mathbf{R}^2 \setminus \{0\}$ is constant.)

3.14 Corollary: *Suppose $n > 2$. If u is positive and harmonic on $\mathbf{R}^n \setminus \{0\}$, then there are nonnegative constants b and c such that $u(x) = b|x|^{2-n} + c$ for all $x \in \mathbf{R}^n \setminus \{0\}$.*

PROOF: On $B \setminus \{0\}$ we may write

$$u(x) = b|x|^{2-n} + v(x)$$

as in Bôcher's Theorem. The function v extends harmonically to all of $\mathbf{R}^n$ by setting $v(x) = u(x) - b|x|^{2-n}$ for $x \in \mathbf{R}^n \setminus B$. Because $\liminf_{x \rightarrow \infty} v(x) \geq 0$, the minimum principle (1.6) implies that v is nonnegative on $\mathbf{R}^n$. By 3.1, v is constant, completing the proof. $\square$

Positive Harmonic Functions on the Ball

At the end of Chapter 2 we briefly discussed how it is possible to define $P[f]$ when f is not continuous, and indicated that it was necessary to do so in order to characterize the bounded harmonic functions on B . A similar idea works for positive harmonic functions on B : Given a positive finite Borel measure μ on S , we can define

$$P[\mu](x) = \int_S P(x, \zeta) d\mu(\zeta)$$

for $x \in B$. The function so defined is positive and harmonic on B , as the reader can check by differentiating under the integral sign or by using the converse to the mean-value property. In Chapter 6 we

will show that every positive harmonic function on B is the Poisson integral of a measure as above. Many important consequences follow from this characterization, among them the result that every positive harmonic function on B has boundary values almost everywhere on S (in a sense to be made precise in Chapter 6); see 6.14 and 6.36.

Exercises

1. Use 3.5 to give another proof of Liouville's Theorem for positive harmonic functions (3.1).
2. Can equality hold in either of the inequalities in Harnack's Inequality for the ball (3.4)?
3. Show that for every multi-index α there exists a constant C_α such that

$$|D^\alpha u(x)| \leq \frac{C_\alpha u(0)}{(1 - |x|)^{|\alpha|+n-1}}$$

for every $x \in B$ and every positive harmonic u on B . Use this to give another proof of Liouville's Theorem for positive harmonic functions.

4. Define s on $\Omega \times \Omega$ by

$$s(x, y) = \sup\{u(y)/u(x) : u \text{ is positive and harmonic on } \Omega\}.$$

Prove that s is continuous on $\Omega \times \Omega$.

5. Suppose Ω is bounded and connected, u is a positive harmonic function on Ω , and α is a multi-index. Show there is a constant C such that

$$|D^\alpha u(a)| \leq \frac{C}{d(a, \partial\Omega)^{|\alpha|+n-1}}$$

for all $a \in \Omega$. Is this true if either of the hypotheses of connectivity or boundedness is dropped?

6. Suppose $z \in H$, u is positive and harmonic on H , and u is bounded on the ray $\{rz : r > 0\}$. Show that u is bounded in the cone $\Gamma_\alpha(0)$ for every $\alpha > 0$.

7. Suppose that $z \in H$, u is positive and harmonic on H , and $u(rz) \rightarrow L$ as $r \rightarrow 0$, where $L \in [0, \infty]$. Show that if $L = \infty$, then u has nontangential limit ∞ at 0. Prove a similar result for the case $L = 0$. Show u need not have nontangential L if $L \in (0, \infty)$.
8. Prove the analogue of Theorem 2.10 for positive harmonic functions.
9. Suppose u is positive and harmonic on H . Show that u has nontangential limit L at 0 if and only if $\lim_{r \rightarrow 0} u(rz) = L$ for every $z \in H$.
10. Show that if pointwise divergence to ∞ occurs in Harnack's Principle (3.8), then the divergence is "uniform" on compact subsets of Ω .
11. Prove that a pointwise convergent sequence of positive harmonic functions on Ω converges uniformly on compact subsets of Ω .
12. Suppose Ω is connected and (u_m) is a sequence of positive harmonic functions on Ω . Show that one of the following statements is true:
 - (a) (u_m) contains a subsequence diverging to ∞ pointwise on Ω ;
 - (b) (u_m) contains a subsequence converging uniformly on compact subsets of Ω .
13. Suppose that u is a radial function in $C^2(B \setminus \{0\})$. Let g be the function on $(0, 1)$ defined by $g(|x|) = u(x)$. Compute Δu in terms of g and its derivatives. Use this to prove that a radial harmonic function on $B \setminus \{0\}$ must be one of the functions in (a) or (b) of Lemma 3.10.
14. Prove that the constant b and the function v in the conclusion of Bôcher's Theorem (3.9) are unique.
15. Suppose $n > 2$. Assume $a \in \Omega$ and u is harmonic on $\Omega \setminus \{a\}$. Show that if u is positive on some punctured ball centered at a , then there exists a nonnegative constant b and a harmonic function v on Ω such that $u(x) = b|x - a|^{2-n} + v(x)$ on $\Omega \setminus \{a\}$.

16. (a) Suppose $n > 2$. Let u be harmonic on $B \setminus \{0\}$. Show that if

$$\liminf_{x \rightarrow 0} u(x)|x|^{n-2} > -\infty,$$

then there exists a function v harmonic on B and a constant b such that $u(x) = b|x|^{2-n} + v(x)$ on B .

(b) Formulate and prove a similar result for $n = 2$.

17. Let $A = \{a_1, a_2, \dots\}$ denote a discrete subset of $\mathbf{R}^n$. Characterize the positive harmonic functions on $\mathbf{R}^n \setminus A$.

CHAPTER 4

The Kelvin Transform

The Kelvin transform performs a role in harmonic function theory analogous to that played by the transformation $f(z) \mapsto f(1/z)$ in holomorphic function theory. For example, it transforms a function harmonic inside the unit sphere into a function harmonic outside the sphere. In this chapter, we introduce the Kelvin transform and use it to solve the Dirichlet problem for the exterior of the unit sphere and to obtain a reflection principle for harmonic functions. Later, we will use the Kelvin transform in the study of isolated singularities of harmonic functions.

Inversion in the Unit Sphere

When studying harmonic functions on unbounded open sets, we will often find it convenient to append the point ∞ to $\mathbf{R}^n$. We topologize $\mathbf{R}^n \cup \{\infty\}$ in the natural way: A set $\omega \subset \mathbf{R}^n \cup \{\infty\}$ is open if it is an open subset of $\mathbf{R}^n$ in the ordinary sense or if $\omega = \{\infty\} \cup (\mathbf{R}^n \setminus E)$, where E is a compact subset of $\mathbf{R}^n$. The resulting topological space is compact and is called the *one-point compactification* of $\mathbf{R}^n$. Via the usual stereographic projection, $\mathbf{R}^n \cup \{\infty\}$ is homeomorphic to the unit sphere in $\mathbf{R}^{n+1}$.

The map $x \mapsto x^*$, where

$$x^* = \begin{cases} x/|x|^2 & \text{if } x \neq 0, \infty \\ 0 & \text{if } x = \infty \\ \infty & \text{if } x = 0 \end{cases}$$

is called the *inversion* of $\mathbf{R}^n \cup \{\infty\}$ relative to the unit sphere. Note that if $x \notin \{0, \infty\}$, then x^* lies on the ray from the origin determined by x , with $|x^*| = 1/|x|$. The reader should verify that the inversion map is continuous, is its own inverse, is the identity on S , and takes a neighborhood of ∞ onto a neighborhood of 0.

For any set $E \subset \mathbf{R}^n \cup \{\infty\}$, we define $E^* = \{x^* : x \in E\}$.

The inversion map preserves the family of spheres and hyperplanes in $\mathbf{R}^n$ (if we adopt the convention that the point ∞ belongs to every hyperplane). To see this, observe that a set $E \subset \mathbf{R}^n$ is a nondegenerate sphere or hyperplane if and only if

$$4.1 \quad E = \{x \in \mathbf{R}^n : a|x|^2 + b \cdot x + c = 0\},$$

where $b \in \mathbf{R}^n$ and a, c are real numbers satisfying $|b|^2 - 4ac > 0$. We easily see that if E has the form 4.1, then E^* has the same form (with the roles of a and c reversed); inversion therefore preserves the family of spheres and hyperplanes as claimed.

Recall that a C^1 -map $\Psi: \Omega \rightarrow \mathbf{R}^n$ is said to be *conformal* if it preserves angles between intersecting curves; this happens if and only if the Jacobian $\Psi'(x)$ is a scalar multiple of an orthogonal transformation for each $x \in \Omega$.

4.2 Proposition: *The inversion $x \rightarrow x^*$ is conformal on $\mathbf{R}^n \setminus \{0\}$.*

PROOF: Set $\Psi(x) = x^* = x/|x|^2$. Fix $y \in \mathbf{R}^n \setminus \{0\}$. Choose an orthogonal transformation T of $\mathbf{R}^n$ such that $Ty = (|y|, 0, \dots, 0)$. Clearly

$$T^{-1} \circ \Psi \circ T = \Psi,$$

so that $T^{-1} \circ \Psi'(T(y)) \circ T = \Psi'(y)$.

Thus to complete the proof we need only show that $\Psi'(T(y))$, which equals $\Psi'(|y|, 0, \dots, 0)$, is a scalar multiple of an orthogonal transformation. However, simple calculus shows that the matrix of $\Psi'(|y|, 0, \dots, 0)$ is diagonal, with $-1/|y|^2$ in the first position and $1/|y|^2$ in the other diagonal positions. The proof of the proposition is complete. $\square$

Motivation and Definition

Suppose E is a compact subset of $\mathbf{R}^n$. If u is harmonic on $\mathbf{R}^n \setminus E$, we naturally regard ∞ as an isolated singularity of u . When should we say that u has a removable singularity at ∞ ? There is an obvious answer when $n = 2$, because here the inversion $x \mapsto x^*$ preserves harmonic functions: If $\Omega \subset \mathbf{R}^2 \setminus \{0\}$ and u is harmonic on Ω , then the function $x \mapsto u(x^*)$ is harmonic on Ω^* . (Note that on $\mathbf{R}^2 = \mathbf{C}$, inversion is the map $z \mapsto 1/\bar{z}$.) When $n = 2$, then, we say that u is harmonic at ∞ provided the function $x \mapsto u(x^*)$ has a removable singularity at 0.

Unfortunately, the inversion map does not preserve harmonic functions when $n > 2$ (consider, for example, $u(x) = |x|^{2-n}$). Nevertheless, there is a transformation involving the inversion that does preserve harmonic functions for all $n \geq 2$; it is called the *Kelvin transformation* in honor of Lord Kelvin who discovered it in the 1840s [11].

We can guess what this transformation is by applying the symmetry lemma to the Poisson kernel. Fixing $\zeta \in S$, recall that $P(\cdot, \zeta)$ is harmonic on $\mathbf{R}^n \setminus \{\zeta\}$ (1.15). By the symmetry lemma, we have

$$|x - \zeta| = ||x|^{-1}x - |\zeta|$$

for all $x \in \mathbf{R}^n \setminus \{0\}$. Applying this to $P(x, \zeta) = (1 - |x|^2)/|x - \zeta|^n$, we easily compute that

$$4.3 \quad P(x, \zeta) = -|x|^{2-n}P(x^*, \zeta)$$

for all $x \in \mathbf{R}^n \setminus \{0, \zeta\}$. The significant fact here is that the right side of 4.3 is a harmonic function of x on $\mathbf{R}^n \setminus \{0, \zeta\}$. Except for the minus sign, the definition of the Kelvin transformation is staring us in the face.

Thus, given a function u defined on a set $E \subset \mathbf{R}^n \setminus \{0\}$, we define the function $K[u]$ on E^* by

$$K[u](x) = |x|^{2-n}u(x^*);$$

the function $K[u]$ is called the *Kelvin transform* of u . Note that when $n = 2$, $K[u](x) = u(x^*)$.

We easily see that $K[K[u]] = u$ for all functions u as above; in other words, the Kelvin transform is its own inverse.

The transform K is also linear: If u, v are functions on E and b, c are constants, then $K[bu + cv] = bK[u] + cK[v]$ on E^* .

We will need the following easily proved fact in the next section: If E is a compact subset of $\mathbf{R}^n \setminus \{0\}$ and (u_m) is a sequence of functions on E , then (u_m) converges uniformly on E if and only if $(K[u_m])$ converges uniformly on E^* .

The Kelvin Transform Preserves Harmonic Functions

We come now to the the crucial property of the Kelvin transform.

4.4 Theorem: *If $\Omega \subset \mathbf{R}^n \setminus \{0\}$, then u is harmonic on Ω if and only if $K[u]$ is harmonic on Ω^* .*

Because the Kelvin transform is its own inverse, we need only show that if u is harmonic on Ω , then $K[u]$ is harmonic on Ω^* . This may be proved by a direct calculation showing $\Delta K[u] = 0$; however, the calculation is long and tedious (see, for example, [13], pages 84–87). We present here a less computational proof that takes advantage of the local expansion (1.27) of a harmonic function into a series of harmonic polynomials.

PROOF OF THEOREM 4.4: We first prove the theorem in the special case where u is harmonic on $\mathbf{R}^n$. Such a function u is, on B , the Poisson integral of $u|_S$. Thus, for $|x| > 1$, we have

$$K[u](x) = \int_S |x|^{2-n} P(x^*, \zeta) u(\zeta) d\sigma(\zeta) = \int_S -P(x, \zeta) u(\zeta) d\sigma(\zeta),$$

the second equality following from 4.3. Differentiating under the integral sign on the right shows $K[u]$ is harmonic on $\{x \in \mathbf{R}^n : |x| > 1\}$. Because products and compositions of real-analytic functions are real analytic (Exercise 22 in Chapter 1), the real analyticity of u on $\mathbf{R}^n$ shows that $K[u]$ is real analytic on $\mathbf{R}^n \setminus \{0\}$. Thus $\Delta K[u]$ —a real-analytic function on $\mathbf{R}^n \setminus \{0\}$ that vanishes on $\{x \in \mathbf{R}^n : |x| > 1\}$ —must be identically zero on $\mathbf{R}^n \setminus \{0\}$ (Theorem 1.23), proving that $K[u]$ is harmonic on $\mathbf{R}^n \setminus \{0\}$.

For the general case, assume u is harmonic on Ω and $a \in \Omega$. Near a , u can be expanded in a uniformly convergent series $\sum u_m$,

where each u_m is a harmonic polynomial; this follows from Theorem 1.27. Because the Kelvin transform is linear and preserves uniform convergence on compact sets, we have $K[u] = \sum K[u_m]$ near a^* , the latter series converging uniformly near a^* . As we just showed, $K[u_m]$ is harmonic on $\mathbf{R}^n \setminus \{0\}$, hence on Ω^* , for each m . Thus $K[u]$ is harmonic near a^* , proving that $K[u]$ is harmonic on Ω^* . $\square$

Harmonicity at Infinity

Because the Kelvin transform preserves harmonic functions, we make the following definition: If $E \subset \mathbf{R}^n$ is compact and u is harmonic on $\mathbf{R}^n \setminus E$, then u is *harmonic at ∞* provided $K[u]$ has a removable singularity at the origin. Notice that in the $n = 2$ case this definition is consistent with our previous discussion.

If u is harmonic at ∞ , then $K[u]$ has a finite limit L at 0; in other words

$$\lim_{x \rightarrow 0} |x|^{2-n} u(x/|x|^2) = L.$$

From this we see that if u is harmonic at ∞ , then $\lim_{x \rightarrow \infty} u(x) = 0$ when $n > 2$, while $\lim_{x \rightarrow \infty} u(x) = L$ when $n = 2$. This observation leads to characterizations of harmonicity at ∞ . We begin with the $n > 2$ case.

4.5 Theorem: *Assume $n > 2$. Suppose u is harmonic on $\mathbf{R}^n \setminus E$, where $E \subset \mathbf{R}^n$ is compact. Then u is harmonic at ∞ if and only if $\lim_{x \rightarrow \infty} u(x) = 0$.*

PROOF: We have just seen that if u is harmonic at ∞ , then $\lim_{x \rightarrow \infty} u(x) = 0$.

Suppose that $\lim_{x \rightarrow \infty} u(x) = 0$. Then $|x|^{n-2} K[u](x) \rightarrow 0$ as $x \rightarrow 0$. By Exercise 2(a) of Chapter 2, $K[u]$ has a removable singularity at 0, which means u is harmonic at ∞ . $\square$

Now we turn to the characterization of harmonicity at ∞ in the $n = 2$ case.

4.6 Theorem: Suppose u is harmonic on $\mathbf{R}^2 \setminus E$, where $E \subset \mathbf{R}^2$ is compact. Then the following are equivalent:

- (a) u is harmonic at ∞ ;
- (b) $\lim_{x \rightarrow \infty} u(x) = L$ for some complex number L ;
- (c) $u(x)/\log|x| \rightarrow 0$ as $x \rightarrow \infty$;
- (d) u is bounded on a deleted neighborhood of ∞ .

PROOF: We have already seen that (a) implies (b).

That (b) implies (c) is trivial.

Suppose now that (c) holds. Then $K[u](x)/\log|x| \rightarrow 0$ as $x \rightarrow 0$. By Exercise 2(b) of Chapter 2, $K[u]$ has a removable singularity at 0. Thus u is harmonic at ∞ , which implies (d).

Finally, if u is bounded on a deleted neighborhood of ∞ , then $K[u](x) = u(x^*)$ is bounded on a deleted neighborhood of 0. Thus (d) implies (a) by Theorem 2.3. $\square$

Boundedness near ∞ is thus equivalent to harmonicity at ∞ when $n = 2$, but not when $n > 2$. We now take up the question of boundedness near ∞ in higher dimensions.

4.7 Theorem: Suppose $n > 2$ and u is harmonic and real valued on $\mathbf{R}^n \setminus E$, where E is compact. Then the following are equivalent:

- (a) u is bounded in a deleted neighborhood of ∞ ;
- (b) u is bounded above or below in a deleted neighborhood of ∞ ;
- (c) $u - c$ is harmonic at ∞ for some constant c ;
- (d) u has a finite limit at ∞ .

PROOF: The implications (a) $\Rightarrow$ (b) and (d) $\Rightarrow$ (a) are trivial. If (c) holds then u has limit c at ∞ by Theorem 4.5; hence (c) $\Rightarrow$ (d). We complete the proof by showing (b) $\Rightarrow$ (c).

Without loss of generality we assume u is positive in a deleted neighborhood of ∞ . Thus $K[u]$ is positive in a deleted neighborhood of 0. By Bôcher's Theorem there is a constant c such that $K[u](x) - c|x|^{2-n}$ extends harmonically across 0. Applying the Kelvin transform shows that $u - c$ is harmonic at ∞ . $\square$

Conditions (a), (c), and (d) of Theorem 4.7 are equivalent without the hypothesis that u is real valued.

The Exterior Dirichlet Problem

In Chapter 1, we solved the Dirichlet problem for the interior of the unit sphere S : Given any $f \in C(S)$, there is a unique function u harmonic on B and continuous on $\bar{B}$ such that $u|_S = f$. To solve the corresponding problem for the exterior of the unit sphere, we define the *exterior Poisson kernel*, denoted P_E , by setting

$$P_E(x, \zeta) = \frac{|x|^2 - 1}{|x - \zeta|^n}$$

for $|x| > 1$ and $\zeta \in S$. Given $f \in C(S)$, we define the exterior Poisson integral $P_E[f]$ by

$$P_E[f](x) = \int_S P_E(x, \zeta) f(\zeta) d\sigma(\zeta)$$

for $|x| > 1$.

4.8 Theorem: Suppose $f \in C(S)$. Then there is a unique function u harmonic on B^* and continuous on $\bar{B}^*$ such that $u|_S = f$. Moreover, $u = P_E[f]$ on $B^* \setminus \{\infty\}$.

REMARK: We are not asserting that there exists a unique continuous u on $\{x \in \mathbf{R}^n : |x| \geq 1\}$, with u harmonic on $\{x \in \mathbf{R}^n : |x| > 1\}$, such that $u = f$ on S . For example, $1 - |x|^{2-n} = 0$ on S when $n > 2$, and $\log|x| = 0$ on S when $n = 2$. The uniqueness assertion in Theorem 4.8 comes from the requirement that u be harmonic on B^* (recall that $\infty \in B^*$).

PROOF OF THEOREM 4.8: Let v denote the solution of the Dirichlet problem for B with boundary data f on S , so that

$$v(x) = \int_S P(x, \zeta) f(\zeta) d\sigma(\zeta)$$

for $x \in B$, and $v = f$ on S . The function $u = K[v]$ is then harmonic on B^* (if we set $u(\infty) = \lim_{x \rightarrow \infty} K[v](x)$), u is continuous on $\bar{B}^*$, and $u|_S = f$.

We have

$$u(x) = \int_S |x|^{2-n} P(x^*, \zeta) f(\zeta) d\sigma(\zeta)$$

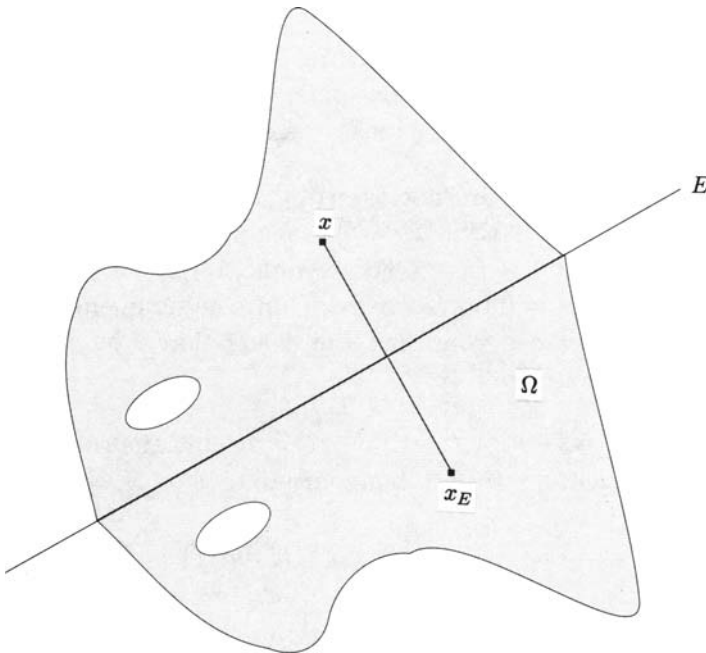
for $|x| > 1$. By 4.3, this gives $u = P_E[f]$ on $B^* \setminus \{\infty\}$, as desired.

The uniqueness of u follows from the maximum principle. $\square$

Symmetry and the Schwarz Reflection Principle

Given a hyperplane E , we say that a pair of points are *symmetric* about E if E is the perpendicular bisector of the line segment joining these points. For each $x \in \mathbb{R}^n$, there exists a unique $x_E \in \mathbb{R}^n$ such that x and x_E are symmetric about E ; we call x_E the *reflection* of x in E . Clearly $(x_E)_E = x$ for every $x \in \mathbb{R}^n$.

We say Ω is *symmetric* about the hyperplane E if $\Omega_E = \Omega$, where $\Omega_E = \{x_E : x \in \Omega\}$.



Ω is symmetric about E .

If T is a translation, dilation, or rotation, then T preserves symmetry about hyperplanes; in other words, if T is any of these maps and E is a hyperplane, then $T(x)$ and $T(x_E)$ are symmetric

about $T(E)$ for all $x \in \mathbf{R}^n$.

Given a hyperplane $E = \{x \in \mathbf{R}^n : b \cdot x = c\}$, where b is a nonzero vector in $\mathbf{R}^n$ and c is a real number, we set $E^+ = \{x \in \mathbf{R}^n : b \cdot x > c\}$; geometrically, E^+ is an open half-space with $\partial E^+ = E$.

We now come to the Schwarz reflection principle for hyperplanes; the reader who has done Exercise 3 in Chapter 2 can probably guess the proof.

4.9 Theorem: *Suppose Ω is symmetric about a hyperplane E . If u is continuous on $\Omega \cap \overline{E^+}$, u is harmonic on $\Omega \cap E^+$, and $u = 0$ on $\Omega \cap E$, then u extends harmonically to Ω .*

PROOF: We may assume that $E = \{(x, y) \in \mathbf{R}^n : y = 0\}$ and that $E^+ = \{(x, y) \in \mathbf{R}^n : y > 0\}$. The function

$$v(x, y) = \begin{cases} u(x, y) & \text{if } (x, y) \in \Omega \text{ and } y \geq 0 \\ -u(x, -y) & \text{if } (x, y) \in \Omega \text{ and } y < 0 \end{cases}$$

is continuous on Ω and satisfies the mean-value property. Hence, by Theorem 1.20, v is a harmonic extension of u to all of Ω . $\square$

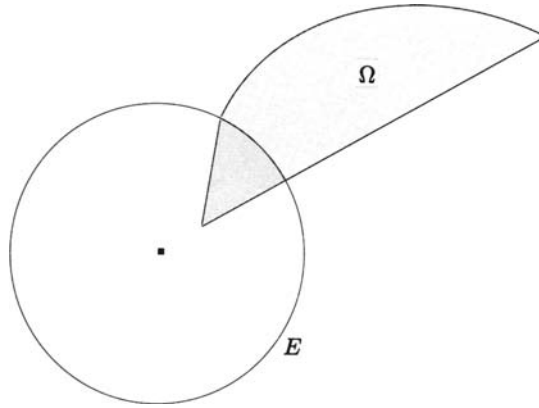
We now extend the notions of symmetry and reflection to spheres. If $E = S$, the unit sphere, then inversion is the natural choice for the reflection map $x \mapsto x_E$. So here we set $x_E = x^*$. More generally, if $E = \partial B(a, r)$, we define

$$4.10 \quad x_E = a + r^2(x - a)^*,$$

and we say that x and x_E are *symmetric* about E . Note that the center of E and the point at infinity are symmetric about E . We say that Ω is *symmetric* about the sphere E if $\Omega_E = \Omega$, where $\Omega_E = \{x_E : x \in \Omega\}$; see 4.11.

We remark in passing that symmetry about a hyperplane can be viewed as a limiting case of symmetry about a sphere; see Exercise 9 of this chapter. (We adopt the convention that $\infty_E = \infty$ when E is a hyperplane.)

Translations, dilations, and rotations clearly preserve symmetry about spheres. The inversion map also preserves symmetry—about spheres as well as hyperplanes—although this is far from obvious. Let us look at a special case we need below. Suppose E is the

**4.11**

Ω is symmetric about E .

sphere with center $(0, \dots, 0, 1)$ and radius 1. Then E contains the origin, so that E^* is a hyperplane; in fact,

$$E^* = \{(x, y) \in \mathbf{R}^n : y = 1/2\} \cup \{\infty\},$$

as the reader can easily check. Assume for the moment that $n = 2$; here we identify $\mathbf{R}^2$ with $\mathbf{C}$, so that inversion is the map $z \mapsto 1/\bar{z}$. Given $z \in \mathbf{C} \setminus \{0\}$, we need to show that z^* and $(z_E)^*$ are symmetric about E^* . A moment's reflection shows that to do this we need only verify that the conjugate of $z^* - i/2$ equals $(z_E)^* - i/2$; this bit of algebra we leave to the reader. To go from $\mathbf{R}^2$ to $\mathbf{R}^n$ with $n > 2$, observe that inversion preserves every linear subspace of $\mathbf{R}^n$. Given $z \in \mathbf{R}^n \setminus \{0\}$, then, we look at the two-dimensional plane determined by 0 , z , and z_E . Because the center of E is on the line determined by z and z_E , this plane contains the $(0, y)$ -axis. We can thus view this plane as $\mathbf{C}$, with $(0, \dots, 0, 1)$ playing the role of i . The proof for $\mathbf{R}^2$ therefore shows that z^* and $(z_E)^*$ are symmetric about E^* in $\mathbf{R}^n$.

We can now prove the Schwarz reflection principle for regions symmetric about spheres.

4.12 Theorem: Suppose Ω is a region symmetric about $\partial B(a, r)$. If u is continuous on $\Omega \cap \bar{B}(a, r)$, u is harmonic on $\Omega \cap B(a, r)$, and $u = 0$ on $\Omega \cap \partial B(a, r)$, then u extends harmonically to Ω .

PROOF: We may assume $a = (0, \dots, 0, 1)$ and $r = 1$; we are then dealing with the sphere E discussed above. Because Ω is sym-

metric about E , Ω^* is symmetric about the hyperplane E^* , as we just showed. Our hypotheses on u now show that the Schwarz reflection principle for hyperplanes (4.9) can be applied to the Kelvin transform of u . Accordingly, $K[u]$ extends to a function v harmonic on Ω^* . Because the Kelvin transform is its own inverse, $K[v]$ extends u harmonically to Ω . $\square$

Let us explicitly identify the harmonic reflection of u across a sphere in the concrete case of S , the unit sphere.

4.13 Theorem: *Suppose Ω is connected and symmetric about S . If u is continuous on $\Omega \cap \bar{B}$, u is harmonic on $\Omega \cap B$, and $u = 0$ on $\Omega \cap S$, then the function v defined on Ω by*

$$v(x) = \begin{cases} u(x) & \text{if } x \in \Omega \cap \bar{B} \\ -K[u](x) & \text{if } x \in \Omega \cap (\mathbf{R}^n \setminus \bar{B}) \end{cases}$$

is the unique harmonic extension of u to Ω .

PROOF: Set $a = (0, \dots, 0, 1)$ and define $w(x) = v(x - a)$; the domain of the function w is then $\Omega + a$, which is symmetric about the sphere E of the previous proof. We will be done if we can show that $K[w]$ has the appropriate reflection property about the hyperplane E^* . What we need to show, then, is that

$$K[w]((x + a)^*) = -K[w]((x^* + a)^*)$$

for all $x \in \Omega$. This amounts to showing

$$|x + a|^{n-2}v(x) = -|x^* + a|^{n-2}v(x^*)$$

for all $x \in \Omega$. By Exercise 1 of this chapter, $|x^* + a|/|x + a| = |x|^{-1}$. We therefore need only show that $v = -K[v]$ on Ω . But this last identity follows easily from the definition of v . $\square$

Exercises

1. Show that if $\zeta \in S$ and $x \in \mathbf{R}^n \setminus \{0\}$, then

$$|x^* + \zeta| = \frac{|x + \zeta|}{|x|}.$$

2. Show that at $x \in \mathbf{R}^n \setminus \{0\}$, the determinant of the Jacobian of the inversion map equals $-1/|x|^{2n}$.
3. Let f be a function of one complex variable that is holomorphic on the complement of some disk. We say that f is *holomorphic at ∞* provided $f(1/z)$ has a removable singularity at 0. Show that the following are equivalent:
- (a) f is holomorphic at ∞ ;
 - (b) f is bounded on a deleted neighborhood of ∞ ;
 - (c) $\lim_{z \rightarrow \infty} f(z)/z = 0$.
4. Assume $\omega \subset \mathbf{R}^n \cup \{\infty\}$ is open. Show that u is harmonic on ω if and only if $K[u]$ is harmonic on ω^* .
5. (a) Show that if $n > 2$, then the only harmonic function on $\mathbf{R}^n \cup \{\infty\}$ is identically zero.
- (b) Prove that the only harmonic functions on $\mathbf{R}^2 \cup \{\infty\}$ are constant.
6. Suppose that u is harmonic and positive on $\mathbf{R}^2 \setminus E$, where E is compact. Characterize the behavior of u near ∞ .
7. Prove that the solution to the exterior Dirichlet problem in Theorem 4.8 is unique.

8. Let E denote the hyperplane $\{x \in \mathbf{R}^n : b \cdot x = c\}$, where b is a nonzero vector in $\mathbf{R}^n$ and c is a real number. For $x \in \mathbf{R}^n$, show that the reflection x_E is given by the formula

$$x_E = x - \frac{2((x \cdot b) - c)b}{|b|^2}.$$

9. Let E denote the hyperplane $\mathbf{R}^{n-1} \times \{0\}$. Fix a point $z = (0, y)$ in the upper half-space. Show that the reflections of z about spheres of radius R centered at $(0, R)$ converge to $z_E = (0, -y)$ as $R \rightarrow \infty$.
10. Suppose E is a sphere of radius r centered at a , with $0 \notin E$. Show that the radius of E^* is $r/|r^2 - |a|^2|$ and the center of E^* is $a/(|a|^2 - r^2)$.
11. Show the inversion map preserves symmetry about spheres and hyperplanes. In other words, if E is a sphere or hyperplane, then x^* and $(x_E)^*$ are symmetric about E^* for all x .
12. Let E be a compact subset of S with non-empty interior relative to S . Prove that there exists a nonconstant bounded harmonic function on $\mathbf{R}^n \setminus E$.

CHAPTER 5

Spherical Harmonics

From the theory of Fourier series on the unit circle we know that when $n = 2$, every $f \in L^2(S)$ has an expansion of the form

$$f(e^{i\theta}) = \sum_{m=-\infty}^{\infty} a_m e^{im\theta},$$

where the sum converges in $L^2(S)$. In this chapter we will see that an analogous expansion is valid for functions $f \in L^2(S)$ when $n > 2$, with objects known as spherical harmonics playing the roles of the exponentials $e^{im\theta}$.

Let $\mathcal{H}_m(\mathbf{R}^n)$ denote the space of all homogeneous harmonic polynomials on $\mathbf{R}^n$ of degree m . For example,

5.1 $p(x, y, z) = 8x^5 - 40x^3y^2 + 15xy^4 - 40x^3z^2 + 30xy^2z^2 + 15xz^4$

is an element of $\mathcal{H}_5(\mathbf{R}^3)$; we have used (x, y, z) in place of (x_1, x_2, x_3) to denote a typical point in $\mathbf{R}^3$.

A *spherical harmonic of degree m* is the restriction to S of an element of $\mathcal{H}_m(\mathbf{R}^n)$. The collection of all spherical harmonics of degree m will be denoted by $\mathcal{H}_m(S)$. For example, take $n = 3$ and consider the function

5.2 $q(x, y, z) = 15x - 70x^3 + 63x^5$

defined for $(x, y, z) \in S$. Is q an element of $\mathcal{H}_5(S)$? Although q appears neither harmonic nor homogeneous of degree 5, note that on S we have

$$q(x, y, z) = 15x(x^2 + y^2 + z^2)^2 - 70x^3(x^2 + y^2 + z^2) + 63x^5.$$

As the reader can easily check, q is the restriction to S of the polynomial p defined in 5.1; therefore $q \in \mathcal{H}_5(S)$. Examples 5.1 and 5.2 were generated using the computer package described in Appendix B.

Orthogonal transformations will play an important role in our study of spherical harmonics. Let $O(n)$ denote the group of orthogonal transformations on $\mathbf{R}^n$. Observe that $\mathcal{H}_m(\mathbf{R}^n)$ is $O(n)$ -invariant, meaning that if $p \in \mathcal{H}_m(\mathbf{R}^n)$ and $T \in O(n)$, then $p \circ T \in \mathcal{H}_m(\mathbf{R}^n)$. It follows that $\mathcal{H}_m(S)$ is $O(n)$ -invariant as well.

Note that $\mathcal{H}_m(\mathbf{R}^n)$, and hence $\mathcal{H}_m(S)$, is a complex vector space.

Let us establish the connection between spherical harmonics and the exponentials when $n = 2$. Suppose $p \in \mathcal{H}_m(\mathbf{R}^2)$ is real valued. Then p is the real part of an entire holomorphic function f whose imaginary part vanishes at the origin. The Cauchy-Riemann equations imply that all derivatives of f except the m^{th} derivative vanish at the origin. Thus f is a monomial, and so

$$p(z) = cz^m + \bar{c}\bar{z}^m$$

for some complex constant c . This implies that $\mathcal{H}_m(\mathbf{R}^2)$ is the complex linear span of $\{z^m, \bar{z}^m\}$. Thus $\mathcal{H}_m(S)$, as a space of functions of the variable $e^{i\theta}$, is the complex linear span of $\{e^{im\theta}, e^{-im\theta}\}$. From this point of view, a Fourier series expansion on the unit circle is the same as an expansion into spherical harmonics.

$$L^2(S) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m(S)$$

The notation in the title of this section comes from Hilbert space theory. If H is a complex Hilbert space, then we write $H = \bigoplus_{m=0}^{\infty} H_m$ when the following three conditions are satisfied:

- (a) H_m is a closed subspace of H for every m .
- (b) H_m is orthogonal to H_k if $m \neq k$.

(c) For every $x \in H$ there exist $x_m \in H_m$ such that

$$x = x_0 + x_1 + \cdots,$$

the sum converging in the norm of H .

When (a), (b), and (c) hold, H is said to be the *direct sum* of the spaces H_m . Note that if this is the case then the expansion in (c) is unique. Note also that if (a) and (b) hold, then (c) holds if and only if the complex linear span of $\bigcup_{m=0}^{\infty} H_m$ is dense in H .

In this section we prove that $L^2(S) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m(S)$, where by $L^2(S)$ we mean the usual Hilbert space of square-integrable functions on S with inner product defined by

$$\langle f, g \rangle = \int_S f \bar{g} d\sigma.$$

The reader can easily see that $L^2(S)$ is the direct sum of the spaces $\mathcal{H}_m(S)$ when $n = 2$ from the L^2 -theory of Fourier series on the circle and the results of the last section.

We now take up (a)–(c) above for all n , with $H = L^2(S)$ and $H_m = \mathcal{H}_m(S)$. Property (a) is easily checked: The finite dimensionality of $\mathcal{H}_m(S)$ shows that $\mathcal{H}_m(S)$ is a closed subspace of $L^2(S)$. The next result establishes (b).

5.3 Theorem: *If $m \neq k$, then $\mathcal{H}_m(S)$ is orthogonal to $\mathcal{H}_k(S)$ in $L^2(S)$.*

PROOF: Let $p \in \mathcal{H}_m(\mathbf{R}^n)$, $q \in \mathcal{H}_k(\mathbf{R}^n)$. Green's identity implies

$$5.4 \quad \int_S (p D_{\mathbf{n}} q - q D_{\mathbf{n}} p) d\sigma = \int_B (p \Delta q - q \Delta p) dV = 0.$$

But for $\zeta \in S$,

$$D_{\mathbf{n}} p(\zeta) = \frac{d}{dr} p(r\zeta)|_{r=1} = \frac{d}{dr} (r^m p(\zeta))|_{r=1} = m p(\zeta).$$

Similarly, $D_{\mathbf{n}} q = k q$ on S . Thus 5.4 implies

$$(k - m) \int_S p q d\sigma = 0.$$

The last integral vanishes because $m \neq k$. This finishes the proof, because $\mathcal{H}_m(\mathbf{R}^n)$ is closed under complex conjugation. $\square$

Proving that condition (c) above is satisfied in our context is more involved. The main step is to show that any polynomial on $\mathbf{R}^n$, when restricted to S , is a sum of spherical harmonics. Let us denote by $\mathcal{P}_m(\mathbf{R}^n)$ the complex vector space of all homogeneous polynomials on $\mathbf{R}^n$ of degree m . We define an inner product $\langle \cdot, \cdot \rangle_m$ on $\mathcal{P}_m(\mathbf{R}^n)$ as follows: For $p(x) = \sum_{|\alpha|=m} a_\alpha x^\alpha$, $q(x) = \sum_{|\alpha|=m} b_\alpha x^\alpha$, set

$$\langle p, q \rangle_m = p(D)[\bar{q}],$$

where $p(D)$ is the partial differential operator $\sum_{|\alpha|=m} a_\alpha D^\alpha$. (Note that we have identified the constant function $p(D)[\bar{q}]$ with its numerical value at all points of $\mathbf{R}^n$.) The reader should verify that

$$\langle p, q \rangle_m = \sum_{|\alpha|=m} \alpha! a_\alpha \bar{b}_\alpha,$$

so that $\langle \cdot, \cdot \rangle_m$ is indeed an inner product on $\mathcal{P}_m(\mathbf{R}^n)$. We learned how this inner product could be used to prove the next theorem from [10], which also provided some of the other ideas in this chapter.

As usual, $[t]$ denotes the largest integer less than or equal to t .

5.5 Theorem: *Every $p \in \mathcal{P}_m(\mathbf{R}^n)$ can be uniquely written in the form*

$$p(x) = p_m(x) + |x|^2 p_{m-2}(x) + \cdots + |x|^{2k} p_{m-2k}(x),$$

where $k = [\frac{m}{2}]$ and $p_{m-2j} \in \mathcal{H}_{m-2j}(\mathbf{R}^n)$ for $j = 0, 1, \dots, k$.

PROOF: Harmonic polynomials that agree on S are identical; thus the uniqueness assertion is clear.

The expansion is clearly possible when $m = 0$ or 1 , because in these cases $\mathcal{P}_m(\mathbf{R}^n) = \mathcal{H}_m(\mathbf{R}^n)$.

The crux of the proof is to show that if $m \geq 2$, then

$$\mathcal{P}_m(\mathbf{R}^n) = \mathcal{H}_m(\mathbf{R}^n) \oplus |x|^2 \mathcal{P}_{m-2}(\mathbf{R}^n)$$

with respect to the inner product $\langle \cdot, \cdot \rangle_m$, where $|x|^2 \mathcal{P}_{m-2}(\mathbf{R}^n) = \{|x|^2 q(x) : q \in \mathcal{P}_{m-2}(\mathbf{R}^n)\}$.

Suppose $p \in \mathcal{P}_m(\mathbf{R}^n)$ is orthogonal to $|x|^2 \mathcal{P}_{m-2}(\mathbf{R}^n)$. Then

$$0 = \langle |x|^2 q, p \rangle_m = q(D)[\Delta \bar{p}] = q(D)[\overline{\Delta p}]$$

for all $q \in \mathcal{P}_{m-2}(\mathbf{R}^n)$. Because $\Delta p \in \mathcal{P}_{m-2}(\mathbf{R}^n)$, the last term above equals $\langle q, \Delta p \rangle_{m-2}$, from which we conclude that Δp is orthogonal to every $q \in \mathcal{P}_{m-2}(\mathbf{R}^n)$. It follows that $\Delta p \equiv 0$, and thus $p \in \mathcal{H}_m(\mathbf{R}^n)$, as desired. Conversely, if $p \in \mathcal{H}_m(\mathbf{R}^n)$, then the preceding argument in reverse shows that p is orthogonal to $|x|^2 \mathcal{P}_{m-2}(\mathbf{R}^n)$.

Having established 5.6, assume $m \geq 2$ and make the induction hypothesis that Theorem 5.5 holds for all polynomials in $\mathcal{P}_j(\mathbf{R}^n)$ for all $j < m$. Let $p \in \mathcal{P}_m(\mathbf{R}^n)$. By 5.6, $p(x) = p_m(x) + |x|^2 q_{m-2}(x)$, where $p_m \in \mathcal{H}_m(\mathbf{R}^n)$ and $q_{m-2} \in \mathcal{P}_{m-2}(\mathbf{R}^n)$. To complete the proof we need only apply the induction hypothesis to q_{m-2} . $\square$

Every polynomial is a sum of homogeneous polynomials, and Theorem 5.5 shows that a homogeneous polynomial of degree m , when restricted to S , is a sum of spherical harmonics of degrees at most m . We have thus established the following corollary.

5.7 Corollary: *If p is a polynomial on $\mathbf{R}^n$ of degree m , then the restriction of p to S is a sum of spherical harmonics of degrees at most m .*

It is now a simple matter to prove the main result of this section.

5.8 Theorem: $L^2(S) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m(S)$.

PROOF: Only property (c) above remains to be verified in the present case, and for this it is sufficient to show that the linear span of $\bigcup_{m=0}^{\infty} \mathcal{H}_m(S)$ is dense in $L^2(S)$. By the Stone-Weierstrass Theorem (see [8], Theorem 7.33), the polynomials are dense in $C(S)$ with respect to the supremum norm. Corollary 5.7 thus implies that finite sums of spherical harmonics are dense in $C(S)$. Because $C(S)$ is dense in $L^2(S)$ and the L^2 -norm is less than or equal to the L^∞ -norm on S , finite sums of spherical harmonics are dense in $L^2(S)$. In other words, the linear span of $\bigcup_{m=0}^{\infty} \mathcal{H}_m(S)$ is dense in $L^2(S)$, as desired. $\square$

Zonal Harmonics

Let us now view $\mathcal{H}_m(S)$ as an inner product space with the $L^2(S)$ -inner product. The notation $\langle \cdot, \cdot \rangle$ and all orthogonality statements will refer to this inner product from now on.

Fix a point $\eta \in S$, and consider the map $\Lambda: \mathcal{H}_m(S) \rightarrow \mathbb{C}$ defined by

$$\Lambda(p) = p(\eta).$$

The map Λ is clearly linear. Thus there exists a unique $Z_\eta \in \mathcal{H}_m(S)$ such that

$$p(\eta) = \langle p, Z_\eta \rangle = \int_S p \overline{Z_\eta} d\sigma$$

for all $p \in \mathcal{H}_m(S)$ (by the self-duality of the finite-dimensional Hilbert space $\mathcal{H}_m(S)$). The spherical harmonic Z_η is called the *zonal harmonic* of degree m with *pole* η . The terminology comes from geometric properties of Z_η that will be explained shortly. At times it will be convenient to write $Z_\eta(\zeta) = Z(\zeta, \eta)$. When the degree m needs to be made explicit, we will write $Z_m(\zeta, \eta)$.

We easily compute Z_m when $n = 2$. Clearly $Z_0 \equiv 1$. For $m > 0$, $\mathcal{H}_m(S)$ is the two-dimensional space spanned by $\{e^{im\theta}, e^{-im\theta}\}$, as we saw earlier. Thus if we fix $e^{i\varphi} \in S$, there are constants $\alpha, \beta \in \mathbb{C}$ such that $Z_m(e^{i\theta}, e^{i\varphi}) = \alpha e^{im\theta} + \beta e^{-im\theta}$. The reproducing property of the zonal harmonic then gives

$$\begin{aligned} \gamma e^{im\varphi} + \delta e^{-im\varphi} &= \int_0^{2\pi} (\gamma e^{im\theta} + \delta e^{-im\theta})(\bar{\alpha} e^{-im\theta} + \bar{\beta} e^{im\theta}) \frac{d\theta}{2\pi} \\ &= \gamma \bar{\alpha} + \delta \bar{\beta} \end{aligned}$$

for every $\gamma, \delta \in \mathbb{C}$. Thus $\alpha = e^{-im\varphi}$ and $\beta = e^{im\varphi}$. We conclude that

$$\mathbf{5.9} \quad Z_m(e^{i\theta}, e^{i\varphi}) = e^{im(\theta-\varphi)} + e^{im(\varphi-\theta)} = 2 \cos m(\theta - \varphi).$$

Later (5.24) we will find an explicit formula for zonal harmonics in higher dimensions.

Returning to the case of arbitrary $n \geq 2$, suppose $p \in \mathcal{H}_m(S)$ is real valued. Then

$$0 = \operatorname{Im} p(\eta) = \operatorname{Im} \int_S p \overline{Z_\eta} d\sigma = - \int_S p (\operatorname{Im} Z_\eta) d\sigma.$$

Taking $p = \text{Im } Z_\eta$ yields

$$\int_S (\text{Im } Z_\eta)^2 d\sigma = 0,$$

which implies $\text{Im } Z_\eta \equiv 0$. We conclude that each Z_η is real valued.

Consider now any orthonormal basis $p_1, \dots, p_{h_m}$ of $\mathcal{H}_m(S)$, where h_m denotes the dimension (over $\mathbb{C}$) of the vector space $\mathcal{H}_m(S)$. By standard Hilbert space theory,

$$5.10 \quad Z_\eta(\zeta) = \sum_{j=1}^{h_m} \langle Z_\eta, p_j \rangle p_j(\zeta) = \sum_{j=1}^{h_m} \overline{p_j(\eta)} p_j(\zeta)$$

for all $\eta, \zeta \in S$. Because Z_η is real valued, 5.10 is unchanged after complex conjugation, and we deduce that

$$Z_\eta(\zeta) = Z_\zeta(\eta)$$

for all $\eta, \zeta \in S$. Note also that 5.10 and Bessel's equality imply

$$5.11 \quad Z_\eta(\eta) = \sum_{j=1}^{h_m} |p_j(\eta)|^2 = \|Z_\eta\|_2^2$$

for all $\eta \in S$, where $\|\cdot\|_2$ denotes the norm in $L^2(S, \sigma)$.

The rotation-invariance of $\mathcal{H}_m(S)$ suggests that 5.11 does not depend on η . Indeed, for any $T \in O(n)$ and any $p \in \mathcal{H}_m(S)$, we have

$$\int_S p Z_{T(\eta)} d\sigma = p(T(\eta)) = \int_S (p \circ T) Z_\eta d\sigma = \int_S p (Z_\eta \circ T^{-1}) d\sigma,$$

the last equality following from the rotation invariance of σ . By the uniqueness of the zonal harmonic $Z_{T(\eta)}$, it follows that

$$5.12 \quad Z_{T(\eta)} = Z_\eta \circ T^{-1}.$$

Evaluating both sides of 5.12 at $T(\eta)$ gives

$$Z_{T(\eta)}(T(\eta)) = Z_\eta(\eta).$$

In other words, the function $\eta \mapsto Z_\eta(\eta)$ is constant on S , as suspected. We can evaluate this constant by integrating the middle term of 5.11 over S , obtaining

$$Z_\eta(\eta) = \int_S \left(\sum_{j=1}^{h_m} |p_j(\eta)|^2 \right) d\sigma(\eta) = h_m$$

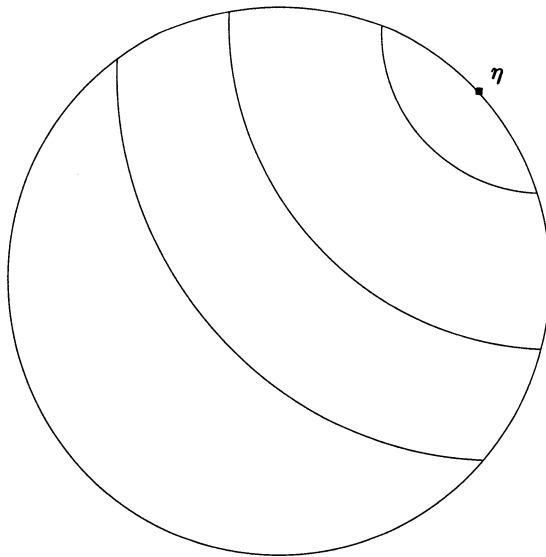
for all $\eta \in S$.

Fittingly, the zonal harmonic Z_η “peaks” at η . To see this, observe that the Schwarz inequality and the last equation imply

$$\mathbf{5.13} \quad |Z_\eta(\zeta)| = |\langle Z_\eta, Z_\zeta \rangle| \leq \|Z_\eta\|_2 \|Z_\zeta\|_2 = h_m = Z_\eta(\eta)$$

for all $\eta, \zeta \in S$.

We now derive a simple geometric property that every zonal harmonic must satisfy. Recall the definition of a “parallel” from cartography: If we identify the surface of the earth with $S \subset \mathbf{R}^3$ so that the north pole is at $(0, 0, 1)$, then a parallel is simply the intersection of S with any plane perpendicular to the z -axis. The notion of a parallel is easily extended to all dimensions: Given $\eta \in S$, we define a *parallel orthogonal* to η to be the intersection of S with any hyperplane perpendicular to η .



Parallels orthogonal to η .

We claim that the zonal harmonic Z_η is constant on each parallel orthogonal to η . To prove this, observe that a function f on

S is constant on parallels orthogonal to η if and only if $f \circ T = f$ for every $T \in O(n)$ with $T(\eta) = \eta$. Using 5.12, we easily verify that $Z_\eta \circ T = Z_\eta$ for every such T , proving our claim. Later (5.30) we will show that scalar multiples of Z_η are the only members of $\mathcal{H}_m(S)$ that are constant on parallels orthogonal to η . This geometric property explains how zonal harmonics came to be named: The term “zonal” refers to the “zones” between parallels orthogonal to the “pole” η .

Our previous decomposition $L^2(S) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m(S)$ has an elegant restatement in terms of zonal harmonics. (Again, this is just the Fourier series decomposition when $n = 2$.)

5.14 Theorem: *If $f \in L^2(S)$, then*

$$f(\eta) = \sum_{m=0}^{\infty} \langle f, Z_m(\cdot, \eta) \rangle$$

in $L^2(S)$.

PROOF: By Theorem 5.8, we have $f = \sum_{m=0}^{\infty} p_m$, where $p_m \in \mathcal{H}_m(S)$ and the infinite sum converges in $L^2(S)$. The proof is completed by noticing that

$$p_m(\eta) = \langle p_m, Z_m(\cdot, \eta) \rangle = \left\langle \sum_{k=0}^{\infty} p_k, Z_m(\cdot, \eta) \right\rangle = \langle f, Z_m(\cdot, \eta) \rangle,$$

where the second equality comes from the orthogonality of spherical harmonics of different degrees. $\square$

We finish this section by computing h_m , the dimension (over $\mathbb{C}$) of the vector space $\mathcal{H}_m(S)$. Note that $\mathcal{H}_m(S)$ and $\mathcal{H}_m(\mathbb{R}^n)$ have the same dimension: If $p \in \mathcal{H}_m(\mathbb{R}^n)$ and $p \equiv 0$ on S , then $p \equiv 0$ on $\mathbb{R}^n$, implying the map $p \mapsto p|_S$ is a vector space isomorphism of $\mathcal{H}_m(\mathbb{R}^n)$ onto $\mathcal{H}_m(S)$.

Because $\mathcal{H}_0(\mathbb{R}^n)$ is the space of constant functions and $\mathcal{H}_1(\mathbb{R}^n)$ is the space of linear functions on $\mathbb{R}^n$, we have $h_0 = 1$ and $h_1 = n$. It is not immediately clear how to proceed for larger values of m . Note, however, that if we let d_m denote the dimension of $\mathcal{P}_m(\mathbb{R}^n)$, then 5.6 implies

$$5.15 \quad d_m = h_m + d_{m-2}$$

when $m \geq 2$.

Computing d_m is an enjoyable combinatorial exercise. Because the monomials $\{x^\alpha : |\alpha| = m\}$ form a basis for $\mathcal{P}_m(\mathbf{R}^n)$, d_m equals the number of distinct multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ with $|\alpha| = m$. We claim that

$$5.16 \quad d_m = \binom{n+m-1}{n-1}.$$

This can be seen as follows: Adding 1 to each α_j , we see that d_m is the number of multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$, with each $\alpha_j > 0$, such that $|\alpha| = n+m$. Consider removing $n-1$ integers from the interval $(0, n+m) \subset \mathbf{R}$. This partitions $(0, n+m)$ into n disjoint open intervals. Letting $\alpha_1, \dots, \alpha_n$ denote the lengths of these intervals, taken in order, we have $\sum_{j=1}^n \alpha_j = n+m$. Each choice of $n-1$ integers from $(0, n+m)$ thus generates a multi-index α with $|\alpha| = n+m$, and each multi-index of degree $n+m$ arises from one and only one such choice. The number of such choices is, of course, $\binom{n+m-1}{n-1}$. We have established 5.16.

From 5.15 it follows that

$$5.17 \quad h_m = \binom{n+m-1}{n-1} - \binom{n+m-3}{n-1}$$

for $m \geq 2$.

The Poisson Kernel Revisited

Every element of $\mathcal{H}_m(S)$ has a unique extension to an element of $\mathcal{H}_m(\mathbf{R}^n)$; given $p \in \mathcal{H}_m(S)$, we will let p denote this extension as well. In particular, the notation $Z_m(\cdot, \zeta)$ will now often refer to the extension of this zonal harmonic to an element of $\mathcal{H}_m(\mathbf{R}^n)$.

Suppose $x \in \mathbf{R}^n$. If $x \neq 0$ and $p \in \mathcal{H}_m(\mathbf{R}^n)$, then

$$\begin{aligned} 5.18 \quad p(x) &= |x|^m p(x/|x|) \\ &= |x|^m \int_S p(\zeta) Z_m(x/|x|, \zeta) d\sigma(\zeta) \\ &= \int_S p(\zeta) Z_m(x, \zeta) d\sigma(\zeta). \end{aligned}$$

We easily check that the first and last terms above also agree when $x = 0$. Note that $Z_m(x, \cdot)$ is a spherical harmonic of degree m for each fixed $x \in \mathbf{R}^n$.

We put these observations to good use in proving our next result, which expresses the Poisson integral of a polynomial in terms of zonal harmonics.

5.19 Theorem: *Let f be a polynomial on $\mathbf{R}^n$ of degree m . Then $P[f]$, the Poisson integral of f , is a polynomial of degree at most m . Moreover,*

$$P[f](x) = \sum_{k=0}^m \int_S Z_k(x, \zeta) f(\zeta) d\sigma(\zeta)$$

for every $x \in B$.

PROOF: We apply 5.7 to write

$$5.20 \quad f = p_0 + p_1 + \cdots + p_m$$

on S , where p_k is a spherical harmonic of degree k . By the uniqueness of the solution to the Dirichlet problem for B , we must have

$$P[f](x) = p_0(x) + p_1(x) + \cdots + p_m(x)$$

for all $x \in B$, now viewing each p_k as a homogeneous harmonic polynomial on $\mathbf{R}^n$. Because spherical harmonics of different degrees are orthogonal, 5.18 shows that

$$p_k(x) = \int_S Z_k(x, \zeta) f(\zeta) d\sigma(\zeta)$$

for all $x \in \mathbf{R}^n$. The desired expansion for $P[f]$ follows. $\square$

Equation 5.20 shows that if f is a polynomial of degree m , then on S , f is orthogonal to all spherical harmonics of higher degree. Thus we could replace the finite sum in 5.19 with an infinite sum. This leads us to the zonal harmonic expansion of the Poisson kernel.

5.21 Theorem: For every $n \geq 2$,

$$5.22 \quad P(x, \zeta) = \sum_{m=0}^{\infty} Z_m(x, \zeta)$$

for all $x \in B$, $\zeta \in S$. The series converges absolutely and uniformly on $K \times S$ for every compact $K \subset B$.

PROOF: For a fixed n , 5.13 and Exercise 7 of this chapter show that there exists a constant C such that

$$|Z_m(x, \zeta)| \leq |x|^m h_m \leq C m^{n-2} |x|^m$$

for all $x \in \mathbf{R}^n$, $\zeta \in S$. The right side of 5.22 therefore has the desired convergence properties.

For a fixed $x \in B$, 5.18 shows that the left and right sides of 5.22 integrate identically against spherical harmonics. Because finite sums of spherical harmonics are dense in $L^2(S)$ (by 5.8), we must have equality in 5.22. $\square$

Formula 5.22 is familiar to us when $n = 2$: From complex analysis (see 1.9) we know that the Poisson kernel for B_2 takes the form

$$P(re^{i\theta}, e^{i\varphi}) = \sum_{-\infty}^{\infty} r^{|k|} e^{ik(\theta-\varphi)} = 1 + \sum_{k=1}^{\infty} r^k 2 \cos k(\theta - \varphi)$$

for all $r \in [0, 1)$ and all $\theta, \varphi \in [0, 2\pi]$. By 5.9, this is exactly the expansion 5.22.

Theorem 5.21 enables us to prove that the homogeneous expansion of an arbitrary harmonic function has the stronger convergence property discussed after Theorem 1.27.

5.23 Corollary: If u is a harmonic function on $B(a, r)$, then there exist $p_m \in \mathcal{H}_m(\mathbf{R}^n)$ such that

$$u(x) = \sum_{m=0}^{\infty} p_m(x - a)$$

for all $x \in B(a, r)$, the series converging absolutely and uniformly on compact subsets of $B(a, r)$.

PROOF: We first assume that u is harmonic on $\bar{B}$. For any $x \in B$, Theorem 5.21 gives

$$u(x) = \int_S P(x, \zeta) u(\zeta) d\sigma(\zeta) = \sum_{m=0}^{\infty} \int_S Z_m(x, \zeta) u(\zeta) d\sigma(\zeta).$$

Letting $p_m(x) = \int_S Z_m(x, \zeta) u(\zeta) d\sigma(\zeta)$ for $x \in \mathbf{R}^n$, observe that $p_m \in \mathcal{H}_m(\mathbf{R}^n)$. As in the proof of Theorem 5.21,

$$|p_m(x)| \leq h_m |x|^m \int_S |u| d\sigma \leq C m^{n-2} |x|^m \int_S |u| d\sigma$$

for all $x \in \mathbf{R}^n$, and thus the series $\sum p_m$ converges absolutely and uniformly to u on compact subsets of B .

After a translation and dilation, the preceding argument shows that if u is harmonic in $B(a, r)$, then u has an expansion of the desired form in each $B(a, s)$, $0 < s < r$. By the uniqueness of homogeneous expansions, all of these expansions are the same, and thus u has the desired expansion in $B(a, r)$. $\square$

An Explicit Formula for Zonal Harmonics

The expansion 5.22 of the Poisson kernel allows us to find an explicit formula for the zonal harmonics.

5.24 Theorem: *Let $x \in \mathbf{R}^n$ and let $\zeta \in S$. Then $Z_m(x, \zeta)$ equals*

$$(n+2m-2) \sum_{k=0}^{[m/2]} (-1)^k \frac{n(n+2) \dots (n+2m-2k-4)}{2^k k! (m-2k)!} (x \cdot \zeta)^{m-2k} |x|^{2k}$$

for each $m > 0$.

PROOF: The function $(1-z)^{-n/2}$ is holomorphic on the unit disk in the complex plane, and so it has a power series expansion

$$5.25 \quad (1-z)^{-n/2} = \sum_{k=0}^{\infty} c_k z^k$$

for $|z| < 1$. We easily compute

$$5.26 \quad c_k = \frac{\left(\frac{n}{2}\right)\left(\frac{n}{2} + 1\right) \dots \left(\frac{n}{2} + k - 1\right)}{k!}.$$

Fix $\zeta \in S$. For $|x|$ small, 5.25 and the binomial formula imply that

$$\begin{aligned} P(x, \zeta) &= (1 - |x|^2)(1 + |x|^2 - 2x \cdot \zeta)^{-n/2} \\ &= (1 - |x|^2) \sum_{k=0}^{\infty} c_k (2x \cdot \zeta - |x|^2)^k \\ &= (1 - |x|^2) \sum_{k=0}^{\infty} c_k \sum_{j=0}^k (-1)^j \binom{k}{j} 2^{k-j} (x \cdot \zeta)^{k-j} |x|^{2j}. \end{aligned}$$

By Theorem 5.21, $Z_m(\cdot, \zeta)$ is equal to the sum of the terms of degree m in the power series representation of $P(\cdot, \zeta)$. Thus the formula above implies that

$$5.27 \quad Z_m(x, \zeta) = q_m(x) - |x|^2 q_{m-2}(x),$$

where q_m and q_{m-2} are the sums of terms of degree m and $m - 2$, respectively, in the double series above. It is easy to see that

$$q_m(x) = \sum_{m/2 \leq k \leq m} c_k (-1)^{m-k} \binom{k}{m-k} 2^{2k-m} (x \cdot \zeta)^{2k-m} |x|^{2m-2k}.$$

Replacing the index k by $m - k$ in this sum shows that

$$q_m(x) = \sum_{k=0}^{[m/2]} c_{m-k} (-1)^k \binom{m-k}{k} 2^{m-2k} (x \cdot \zeta)^{m-2k} |x|^{2k}.$$

Using 5.26, the last equation becomes

$$q_m(x) = \sum_{k=0}^{[m/2]} (-1)^k \frac{n(n+2) \dots (n+2m-2k-2)}{2^k k! (m-2k)!} (x \cdot \zeta)^{m-2k} |x|^{2k}.$$

By replacing m by $m - 2$, we obtain a formula for q_{m-2} . In that formula, replace the index k by $k - 1$, and then combine terms in 5.27 to complete the proof. $\square$

Note that for $x \in S$, the expansion in 5.24 shows that $Z_m(x, \zeta)$ is a function of $x \cdot \zeta$. We could have predicted this in advance by recalling that on S , $Z_m(\cdot, \zeta)$ is constant on parallels orthogonal to ζ .

The formula for zonal harmonics (5.24) can be combined with Theorem 5.19 to calculate explicitly the Poisson integral of any polynomial. This procedure has been implemented by the software described in Appendix B. For example, the software shows that if $n = 5$, then the Poisson integral of x_1^3 is

$$\frac{3x_1 + 4x_1^3 - 3x_1x_2^2 - 3x_1x_3^2 - 3x_1x_4^2 - 3x_1x_5^2}{7}.$$

A Geometric Characterization of Zonal Harmonics

In this section we show that scalar multiples of Z_η are the only spherical harmonics of degree m that are constant on parallels orthogonal to η . The following two lemmas will be helpful in proving this theorem.

5.28 Lemma: *If f is real analytic and radial on $\mathbf{R}^n$, then there exist constants $c_m \in \mathbf{C}$ such that*

$$f(x) = \sum_{m=0}^{\infty} c_m |x|^{2m}$$

for all x near 0.

PROOF: Assume first that $f \in \mathcal{P}_m(\mathbf{R}^n)$ and that f is not identically 0. Because f is radial, it has a constant value $c \neq 0$ on S , which implies that

$$f(x) = c|x|^m$$

for all $x \in \mathbf{R}^n$. If m were odd, we would have $f(-x) = (-1)^m f(x) = -f(x)$ for all $x \in \mathbf{R}^n$, violating the equation above. Thus m is even, proving that f has the desired form in this case.

Now suppose that f is real analytic and radial, and that $\sum p_m$ is the homogeneous expansion of f near 0. Let $T \in O(n)$. Because f is radial, $f = f \circ T$, which gives $\sum p_m = \sum p_m \circ T$ near 0. Since p_m is a homogeneous polynomial of degree m , the same is true of $p_m \circ T$, so that $p_m = p_m \circ T$ for every m by the uniqueness of the homogeneous expansion of f . This is true for every $T \in O(n)$, and therefore each p_m is radial. The result in the previous paragraph now completes the proof. $\square$

5.29 Lemma: *Suppose that u is harmonic on $\mathbf{R}^n$ and that $u(\cdot, y)$ is radial on $\mathbf{R}^{n-1}$ for each $y \in \mathbf{R}$. Suppose further that $u(0, y) = 0$ for all $y \in \mathbf{R}$. Then $u \equiv 0$.*

PROOF: Recall that the power series of a function harmonic on $\mathbf{R}^n$ converges everywhere on $\mathbf{R}^n$ (Exercise 24 in Chapter 1). Because u is real analytic on $\mathbf{R}^n$ and each $u(\cdot, y)$ is radial on $\mathbf{R}^{n-1}$, Lemma 5.28 implies that the expansion of u takes the form

$$u(x, y) = \sum_{m=0}^{\infty} c_m(y) |x|^{2m},$$

where each c_m is a real-analytic function of y . Because u is harmonic, we obtain

$$\begin{aligned} 0 &= \Delta u(x, y) \\ &= \sum_{m=0}^{\infty} c_m''(y) |x|^{2m} + \sum_{m=1}^{\infty} \alpha_m c_m(y) |x|^{2(m-1)} \\ &= \sum_{m=0}^{\infty} [c_m''(y) + \alpha_{m+1} c_{m+1}(y)] |x|^{2m}, \end{aligned}$$

where $\alpha_m = 2m(2m + n - 3)$. Looking at the last series, we see that each term in brackets vanishes. Because $c_0(y) = u(0, y) \equiv 0$, we easily verify by induction that each c_m is identically zero. Thus $u \equiv 0$, as desired. $\square$

Let $\mathbf{N}$ denote the north pole $(0, \dots, 0, 1)$. We can now characterize the zonal harmonics geometrically.

5.30 Theorem: *A spherical harmonic of degree m is constant on parallels orthogonal to η if and only if it is a constant multiple of $Z_m(\cdot, \eta)$.*

PROOF: We have already seen that $Z_m(\cdot, \eta)$ is constant on parallels orthogonal to η .

For the converse, we may assume $m \geq 1$. For convenience we first treat the case $\eta = \mathbf{N}$. So suppose $p \in \mathcal{H}_m(\mathbf{R}^n)$ is constant on parallels orthogonal to $\mathbf{N}$. Fixing any $T \in O(n-1)$, we then have

$$p(Tx, y) = p(x, y)$$

for all $(x, y) \in S$, and hence for all $(x, y) \in \mathbf{R}^n$. Since this is true for all $T \in O(n-1)$, $p(\cdot, y)$ is radial on $\mathbf{R}^{n-1}$ for each $y \in \mathbf{R}$. In particular, $Z_m(\cdot, y, \mathbf{N})$, regarded as an element of $\mathcal{H}_m(\mathbf{R}^n)$, is radial on $\mathbf{R}^{n-1}$ for each $y \in \mathbf{R}$.

Now choose c such that $p(\mathbf{N}) = cZ_m(\mathbf{N}, \mathbf{N})$, and define

$$u = p - cZ_m(\cdot, \mathbf{N}).$$

Then u is harmonic on $\mathbf{R}^n$, $u(\cdot, y)$ is radial on $\mathbf{R}^{n-1}$ for each $y \in \mathbf{R}$, and $u(0, y) = u(y\mathbf{N}) = y^m u(\mathbf{N}) = 0$ for every $y \in \mathbf{R}$. By Lemma 5.29, $u \equiv 0$. Thus p is a constant multiple of $Z_m(\cdot, \mathbf{N})$, as desired.

For the general $\eta \in S$, choose $T \in O(n)$ such that $T(\mathbf{N}) = \eta$. If $p \in \mathcal{H}_m(S)$ is constant on parallels orthogonal to η , then $p \circ T$ is constant on parallels orthogonal to $\mathbf{N}$. Hence $p \circ T$ is a constant multiple of $Z_m(\cdot, \mathbf{N}) = Z_{\mathbf{N}}$, which implies, by 5.12, that p is a constant multiple of $Z_{\mathbf{N}} \circ T^{-1} = Z_{T(\mathbf{N})} = Z_{\eta}$. $\square$

Spherical Harmonics Via Differentiation

Assuming $n > 2$, let us compute a few derivatives of the function $|x|^{2-n}$. We find the answer is always of the same form—a polynomial divided by a power of $|x|$. For example,

$$D_1^2(|x|^{2-n}) = \frac{(2-n)(|x|^2 - nx_1^2)}{|x|^{n+2}}.$$

Notice here that the polynomial in the numerator is harmonic. This is no accident—differentiating $|x|^{2-n}$ exactly k times will always leave us with a homogeneous harmonic polynomial of degree k divided by $|x|^{n-2+2k}$. Moreover, all homogeneous harmonic polynomials arise as numerators in this manner. The main goal of this section is to prove these claims.

The Kelvin transform will play a key role here. To see why, observe that the Kelvin transform applied to the example in the first paragraph leaves us with the harmonic polynomial in the numerator. This indicates how we will obtain homogeneous harmonic polynomials: We first differentiate $|x|^{2-n}$, and then we apply the Kelvin transform. Note that if p is a homogeneous polynomial on $\mathbf{R}^n$ of degree m , then $K[p](x) = |x|^{2-n-2m}p(x)$.

We start with a strange but useful identity involving differentiation and the Kelvin transform.

5.31 Lemma ($n > 2$): Suppose $m > 0$. If $p \in \mathcal{H}_m(\mathbf{R}^n)$, then

$$K[p] = \frac{1}{m(4-n-2m)} \sum_{j=1}^n D_j K[D_j p].$$

PROOF: Because $D_j p \in \mathcal{H}_{m-1}(\mathbf{R}^n)$, we have $K[D_j p](x) = |x|^{4-n-2m} D_j p(x)$. Thus

$$\begin{aligned} \sum_{j=1}^n D_j K[D_j p](x) &= \sum_{j=1}^n \frac{(4-n-2m)x_j D_j p(x) + |x|^2 D_j^2 p(x)}{|x|^{n+2m-2}} \\ &= \frac{(4-n-2m)x \cdot \nabla p(x) + |x|^2 \Delta p(x)}{|x|^{n+2m-2}} \\ &= \frac{(4-n-2m)x \cdot \nabla p(x)}{|x|^{n+2m-2}} \\ &= \frac{m(4-n-2m)p(x)}{|x|^{n+2m-2}} \\ &= m(4-n-2m)K[p](x), \end{aligned}$$

where we have used the identity $x \cdot \nabla p(x) = mp(x)$ from Exercise 20 in Chapter 1. $\square$

To motivate our next result, consider the harmonic homogeneous polynomial $p(x) = x_1^2 - x_2^2$. We easily calculate that

$$K[p(D)]|x|^{2-n} = (n-2)n(x_1^2 - x_2^2).$$

Notice that the right side of this equation is a constant multiple of p . The next theorem shows that this always happens when p is a homogeneous harmonic polynomial.

5.32 Theorem ($n > 2$): Let $p \in \mathcal{H}_m(\mathbf{R}^n)$. Then

$$p = c_m K[p(D)]|x|^{2-n},$$

where $c_m = \prod_{j=1}^m (4-n-2j)^{-1}$.

PROOF: The proof will be by induction on m . The result is clearly true for $m = 0$ (c_0 , being an empty product, is equal to 1). Suppose $m > 0$ and Theorem 5.32 holds for $m - 1$. Let $p \in \mathcal{H}_m(\mathbf{R}^n)$. Note that $D_j p \in \mathcal{H}_{m-1}(\mathbf{R}^n)$. Thus, by Lemma 5.31 and the induction hypothesis, we have

$$\begin{aligned}
 K[p] &= \frac{1}{m(4-n-2m)} \sum_{j=1}^n D_j K[D_j p] \\
 &= \frac{c_{m-1}}{m(4-n-2m)} \sum_{j=1}^n D_j K[K[(D_j p)(D)|x|^{2-n}]] \\
 &= \frac{c_m}{m} \sum_{j=1}^n D_j [(D_j p)(D)|x|^{2-n}] \\
 &= \frac{c_m}{m} D \cdot (\nabla p)(D)|x|^{2-n} \\
 &= c_m p(D)|x|^{2-n},
 \end{aligned}$$

as desired. □

We can now prove the main result of this section.

5.33 Theorem ($n > 2$): $\mathcal{H}_m(\mathbf{R}^n)$ is the linear span of

$$\{K[D^\alpha |x|^{2-n}] : |\alpha| = m\},$$

and $\mathcal{H}_m(S)$ is the linear span of

$$\{(D^\alpha |x|^{2-n})|_S : |\alpha| = m\}.$$

PROOF: Denote the linear span of $\{K[D^\alpha |x|^{2-n}] : |\alpha| = m\}$ by $\mathcal{L}$. We easily see that $\mathcal{H}_m(\mathbf{R}^n) \subset \mathcal{L}$ from 5.32 and the linearity of K . For the other inclusion, let α denote a multi-index of degree m . Writing

$$x^\alpha = p_m(x) + |x|^2 p_{m-2}(x) + \cdots + |x|^{2k} p_{m-2k}(x)$$

as in 5.5, note that $D^\alpha |x|^{2-n} = p_m(D)|x|^{2-n}$ (because $|x|^{2-n}$ is harmonic). Since $p_m \in \mathcal{H}_m(\mathbf{R}^n)$, we have $K[D^\alpha |x|^{2-n}] \in \mathcal{H}_m(\mathbf{R}^n)$ by 5.32, giving $\mathcal{L} \subset \mathcal{H}_m(\mathbf{R}^n)$.

Because $K[f] = f$ for every function f defined on S , the second assertion of the theorem follows from the first assertion. $\square$

We have concentrated in this section on the case $n > 2$. As the reader might expect, analogous results hold when $n = 2$ if $|x|^{2-n}$ is replaced by $\log|x|$. See Exercise 17 of this chapter for the two-dimensional analogue of Theorem 5.32.

Explicit Bases for $\mathcal{H}_m(\mathbf{R}^n)$ and $\mathcal{H}_m(S)$

Early in the chapter we saw that $\{z^m, \bar{z}^m\}$ is a basis for $\mathcal{H}_m(\mathbf{R}^2)$ and $\{e^{im\theta}, e^{-im\theta}\}$ is a basis for $\mathcal{H}_m(S)$. From the spanning sets in Theorem 5.33 we now identify bases for $\mathcal{H}_m(\mathbf{R}^n)$ and $\mathcal{H}_m(S)$ when $n > 2$.

5.34 Theorem ($n > 2$): *The set*

$$\{K[D^\alpha|x|^{2-n}] : |\alpha| = m, \alpha_1 = 0 \text{ or } 1\}$$

is a vector space basis for $\mathcal{H}_m(\mathbf{R}^n)$, and the set

$$\{D^\alpha|x|^{2-n} : |\alpha| = m, \alpha_1 = 0 \text{ or } 1\}$$

is a vector space basis for $\mathcal{H}_m(S)$.

PROOF: Let $\mathcal{B} = \{K[D^\alpha|x|^{2-n}] : |\alpha| = m, \alpha_1 = 0 \text{ or } 1\}$. We first show that $\mathcal{B}$ spans $\mathcal{H}_m(\mathbf{R}^n)$. For this we need only show that $K[D^\alpha|x|^{2-n}]$ is in the span of $\mathcal{B}$ for every multi-index α of degree m . So suppose α is a multi-index of degree m . If α_1 is 0 or 1, then $K[D^\alpha|x|^{2-n}]$ is in $\mathcal{B}$ by definition. Now we use induction on α_1 . Suppose that $\alpha_1 > 1$ and that $K[D^\beta|x|^{2-n}]$ is in the span of $\mathcal{B}$ for all multi-indices β of degree m whose first components are less than α_1 . Because $\Delta|x|^{2-n} \equiv 0$, we have

$$\begin{aligned} K[D^\alpha|x|^{2-n}] &= K[D_1^{\alpha_1-2}D_2^{\alpha_2} \dots D_n^{\alpha_n}(D_1^2|x|^{2-n})] \\ &= -K[D_1^{\alpha_1-2}D_2^{\alpha_2} \dots D_n^{\alpha_n}(\sum_{j=2}^n D_j^2|x|^{2-n})] \\ &= -\sum_{j=2}^n K[D_1^{\alpha_1-2}D_2^{\alpha_2} \dots D_n^{\alpha_n}(D_j^2|x|^{2-n})]. \end{aligned}$$

By our induction hypothesis, each of the summands in the last line is in the span of $\mathcal{B}$, and therefore $K[D^\alpha|x|^{2-n}]$ is in the span of $\mathcal{B}$. We conclude that $\mathcal{B}$ spans $\mathcal{H}_m(\mathbf{R}^n)$.

To complete the proof that $\mathcal{B}$ is a basis for $\mathcal{H}_m(\mathbf{R}^n)$, we show that the cardinality of $\mathcal{B}$ is at most the dimension of $\mathcal{H}_m(\mathbf{R}^n)$. We have $\mathcal{B} = \{K[D^\alpha|x|^{2-n}]\}$, where α ranges over multi-indices of length m that are not of the form $(\beta_1 + 2, \beta_2, \dots, \beta_n)$ with $|\beta| = m - 2$. Therefore the cardinality of $\mathcal{B}$ is at most

$$\#\{\alpha : |\alpha| = m\} - \#\{\beta : |\beta| = m - 2\},$$

where $\#$ denotes cardinality. But from 5.15 we know that this difference equals the dimension of $\mathcal{H}_m(\mathbf{R}^n)$.

Having shown that $\mathcal{B}$ is a basis for $\mathcal{H}_m(\mathbf{R}^n)$, we see that $\{D^\alpha|x|^{2-n} : |\alpha| = m, \alpha_1 = 0 \text{ or } 1\}$ is a basis for $\mathcal{H}_m(S)$ as in the proof of Theorem 5.33. $\square$

The software described in Appendix B uses Theorem 5.34 to construct bases for $\mathcal{H}_m(\mathbf{R}^n)$ and $\mathcal{H}_m(S)$. For example, this software produces the following vector space basis for $\mathcal{H}_4(\mathbf{R}^3)$:

$$\begin{aligned} &\{ \ 3|x|^4 - 30|x|^2x_2^2 + 35x_2^4, \\ &\quad 3|x|^2x_2x_3 - 7x_2^3x_3, \\ &\quad |x|^4 - 5|x|^2x_2^2 - 5|x|^2x_3^2 + 35x_2^2x_3^2, \\ &\quad 3|x|^2x_2x_3 - 7x_2x_3^3, \\ &\quad 3|x|^4 - 30|x|^2x_3^2 + 35x_3^4, \\ &\quad 3|x|^2x_1x_2 - 7x_1x_2^3, \\ &\quad |x|^2x_1x_3 - 7x_1x_2^2x_3, \\ &\quad |x|^2x_1x_2 - 7x_1x_2x_3^2, \\ &\quad 3|x|^2x_1x_3 - 7x_1x_3^3 \}. \end{aligned}$$

Exercises

1. Use Corollary 5.7 and the Stone-Weierstrass Theorem to show that the Dirichlet problem for B has a solution.
2. Let f be a polynomial on $\mathbf{R}^n$. Prove that $P_E[f|_S]$, the exterior Poisson integral of $f|_S$ (see Chapter 4), extends to a function that is harmonic on $\mathbf{R}^n \setminus \{0\}$.
3. Suppose that f is a homogeneous polynomial on $\mathbf{R}^n$ of even (respectively, odd) degree. Prove that $P[f]$ is a polynomial consisting only of terms of even (respectively, odd) degree.
4. Given a polynomial f on $\mathbf{R}^n$, how would you go about determining whether or not $f|_S$ is a spherical harmonic?
5. From Pascal's triangle we know $\binom{N+1}{M} = \binom{N}{M} + \binom{N}{M-1}$. Use this and 5.17 to show that

$$h_m = \binom{n+m-2}{n-2} + \binom{n+m-3}{n-2}$$

for $m > 0$.

6. Prove that $h_0 < h_1 < h_2 < \cdots$ when $n > 2$.
7. Prove that for a fixed n ,

$$\frac{h_m}{m^{n-2}} \rightarrow \frac{2}{(n-2)!}$$

as $m \rightarrow \infty$.

8. Give an example of a real-analytic function in B whose homogeneous expansion (about 0) does not converge in all of B . (Compare this with Corollary 5.23.)

9. Define $P_M(x, \zeta) = \sum_{m=0}^M Z_m(x, \zeta)$. Show that for fixed $\zeta \in S$,

$$\inf_{x \in B} P_M(x, \zeta) \rightarrow -\infty \quad \text{as } M \rightarrow \infty,$$

even though for each fixed $x \in B$, $P_M(x, \zeta) \rightarrow P(x, \zeta) > 0$ as $M \rightarrow \infty$ (by 5.22).

10. Fix a positive integer m . By Theorem 5.24, there is a polynomial q of one variable such that $Z_m(\eta, \zeta) = q(\eta \cdot \zeta)$ for all $\eta, \zeta \in S$. Prove that if n is even, then each coefficient of q is an integer.
11. Suppose u is harmonic on $\mathbf{R}^n$ and u is constant on parallels orthogonal to $\eta \in S$. Show that there exist $c_0, c_1, \dots \in \mathbf{C}$ such that

$$u(x) = \sum_{m=0}^{\infty} c_m Z_m(x, \eta)$$

for all $x \in \mathbf{R}^n$.

12. Suppose that u is harmonic on $\bar{B}$, u is constant on parallels orthogonal to $\eta \in S$, and $u(r\eta) = 0$ for infinitely many $r \in [-1, 1]$. Prove that $u \equiv 0$ in B .
13. Show that u need not vanish identically in Exercise 12 if “harmonic on $\bar{B}$ ” is replaced by “continuous on $\bar{B}$ and harmonic on B ”. (Suggestion: Set $q_m(x) = Z_m(x, \eta)/h_m$ and consider a sum of the form $\sum_{k=1}^{\infty} (-1)^k c_k q_{m_k}(x)$, where the coefficients c_k are positive and summable and the integers m_k are widely spaced.)
14. Show that there exists a nonconstant harmonic function u on $\mathbf{R}^2$ that is constant on parallels orthogonal to $e^{i\theta}$ as well as on parallels orthogonal to $e^{i\varphi}$ if and only if $\theta - \varphi$ is a rational multiple of π .
15. Suppose $n > 2$ and $\zeta, \eta \in S$. Under what conditions can a function on S be constant on parallels orthogonal to ζ as well as on parallels orthogonal to η ?

16. Prove that if $p \in \mathcal{H}_m(\mathbf{R}^n)$, then

$$D_j K[p] = K[|x|^2 D_j p + (2 - n - 2m)x_j p]$$

for $1 \leq j \leq n$.

17. Prove that if $m > 0$ and $p \in \mathcal{H}_m(\mathbf{R}^2)$, then

$$p(x) = c_m K[p(D) \log |x|],$$

where $c_m = \frac{(-1)^{m-1}}{2^{m-1}(m-1)!}$. (This is the analogue for $n = 2$ of Theorem 5.32.)

18. *Generalized Dirichlet Problem:* Show that if f and g are polynomials on $\mathbf{R}^n$, then there is a polynomial u with $u|_S = f|_S$ and $\Delta u = g$. (The software described in Appendix B can find u explicitly.)

CHAPTER 6

Harmonic Hardy Spaces

Poisson Integrals of Measures

In Chapter 1 we defined the Poisson integral of a function $f \in C(S)$ to be the function $P[f]$ defined on B by

$$6.1 \quad P[f](x) = \int_S P(x, \zeta) f(\zeta) d\sigma(\zeta).$$

We now extend this definition: For μ a complex Borel measure on S , the Poisson integral of μ , denoted $P[\mu]$, is the function on B defined by

$$6.2 \quad P[\mu](x) = \int_S P(x, \zeta) d\mu(\zeta).$$

Differentiating under the integral sign in 6.2, we see that $P[\mu]$ is harmonic in B .

The set of complex Borel measures on S will be denoted by $M(S)$. The total variation norm of $\mu \in M(S)$ will be denoted by $\|\mu\|$. Recall that $M(S)$ is a Banach space under the total variation norm. By the Riesz Representation Theorem, if we identify $\mu \in M(S)$ with the linear functional Λ_μ on $C(S)$ given by

$$\Lambda_\mu(f) = \int_S f d\mu,$$

then $M(S)$ is isometrically isomorphic to the dual space of $C(S)$. (A good source for these results is [9].)

We will also deal with the Banach spaces $L^p(S)$, $1 \leq p \leq \infty$. When $p \in [1, \infty)$, $L^p(S)$ consists of the Borel measurable functions f on S for which

$$\|f\|_p = \left(\int_S |f|^p d\sigma \right)^{1/p} < \infty;$$

$L^\infty(S)$ consists of the Borel measurable functions f on S for which $\|f\|_\infty < \infty$, where $\|f\|_\infty$ denotes the essential supremum norm on S with respect to σ . The number $q \in [1, \infty]$ is said to be *conjugate* to p if $1/p + 1/q = 1$. If $1 \leq p < \infty$ and q is conjugate to p , then $L^q(S)$ is the dual space of $L^p(S)$. Here we identify $g \in L^q(S)$ with the linear functional Λ_g on $L^p(S)$ defined by

$$\Lambda_g(f) = \int_S fg d\sigma.$$

Note that because σ is a finite measure on S , $L^p(S) \subset L^1(S)$ for all $p \in [1, \infty]$. Recall also that $C(S)$ is dense in $L^p(S)$ for $1 \leq p < \infty$.

It is natural to identify each $f \in L^1(S)$ with the measure $\mu_f \in M(S)$ defined on Borel sets $E \subset S$ by

$$\mathbf{6.3} \quad \mu_f(E) = \int_E f d\sigma.$$

Shorthand for 6.3 is the expression $d\mu_f = f d\sigma$. The map $f \mapsto \mu_f$ is a linear isometry of $L^1(S)$ into $M(S)$. We will often identify functions in $L^1(S)$ as measures in this manner without further comment.

For $f \in L^1(S)$, we will write $P[f]$ in place of $P[\mu_f]$. Here one could also try to define $P[f]$ as in 6.1. Fortunately the two definitions agree, because if φ is a bounded Borel measurable function on S (in particular, if $\varphi = P(x, \cdot)$), then $\int_S \varphi d\mu_f = \int_S (\varphi f) d\sigma$. Our notation is thus consistent with that defined previously for continuous functions on S .

Throughout this chapter, when given a function u on B , the notation u_r will refer to the function on S defined by $u_r(\zeta) = u(r\zeta)$; here, of course, $0 \leq r < 1$.

We know that when $f \in C(S)$, $P[f]$ has a continuous extension to $\bar{B}$. What can be said of the more general Poisson integrals defined above? We begin to answer this question in the next two theorems.

6.4 Theorem: *The following growth estimates apply to Poisson integrals:*

- (a) *If $\mu \in M(S)$ and $u = P[\mu]$, then $\|u_r\|_1 \leq \|\mu\|$ for all $r \in [0, 1)$.*
- (b) *Suppose $1 \leq p \leq \infty$. If $f \in L^p(S)$ and $u = P[f]$, then $\|u_r\|_p \leq \|f\|_p$ for all $r \in [0, 1)$.*

PROOF: The identity

$$6.5 \quad P(r\eta, \zeta) = P(r\zeta, \eta),$$

valid for all $\eta, \zeta \in S$ and all $r \in [0, 1)$, will be used to prove both (a) and (b).

To prove (a), let $\mu \in M(S)$ and set $u = P[\mu]$. For $\eta \in S$ and $r \in [0, 1)$,

$$|u(r\eta)| \leq \int_S P(r\eta, \zeta) d|\mu|(\zeta),$$

where $|\mu|$ denotes the total variation measure associated with μ . Fubini's theorem and 6.5 then give

$$\begin{aligned} \|u_r\|_1 &= \int_S |u(r\eta)| d\sigma(\eta) \\ &\leq \int_S \int_S P(r\eta, \zeta) d|\mu|(\zeta) d\sigma(\eta) \\ &= \int_S \int_S P(r\zeta, \eta) d\sigma(\eta) d|\mu|(\zeta) \\ &= \|\mu\|. \end{aligned}$$

For (b), assume first that $1 \leq p < \infty$. Let $f \in L^p(S)$ and set $u = P[f]$. Then

$$|u(r\eta)| \leq \int_S P(r\eta, \zeta) |f(\zeta)| d\sigma(\zeta).$$

By Jensen's inequality,

$$|u(r\eta)|^p \leq \int_S P(r\eta, \zeta) |f(\zeta)|^p d\sigma(\zeta).$$

Integrating this last expression over S , an argument similar to that given for (a) yields $\|u_r\|_p \leq \|f\|_p$, as desired.

The case $f \in L^\infty(S)$ is the easiest. With $u = P[f]$, we have

$$\begin{aligned} |u(r\eta)| &\leq \int_S P(r\eta, \zeta) |f(\zeta)| d\sigma(\zeta) \\ &\leq \|f\|_\infty \int_S P(r\eta, \zeta) d\sigma(\zeta) \\ &= \|f\|_\infty. \end{aligned} \quad \square$$

If $f \in C(S)$ and $u = P[f]$, we know that $u_r \rightarrow f$ in $C(S)$ as $r \rightarrow 1$. This fact and Theorem 6.4 enable us to prove the following result on L^p -convergence.

6.6 Theorem: Suppose $1 \leq p < \infty$. If $f \in L^p(S)$ and $u = P[f]$, then $\|u_r - f\|_p \rightarrow 0$ as $r \rightarrow 1$.

PROOF: Let $p \in [1, \infty)$, let $f \in L^p(S)$, and set $u = P[f]$. Fix $\varepsilon > 0$, and choose $g \in C(S)$ with $\|f - g\|_p < \varepsilon$. Setting $v = P[g]$, we have

$$\|u_r - f\|_p \leq \|u_r - v_r\|_p + \|v_r - g\|_p + \|g - f\|_p.$$

Now $(u_r - v_r) = (P[f - g])_r$, hence $\|u_r - v_r\|_p < \varepsilon$ by Theorem 6.4. Note also that $\|v_r - g\|_p \leq \|v_r - g\|_\infty$. Thus

$$\|u_r - f\|_p < \|v_r - g\|_\infty + 2\varepsilon.$$

Because $g \in C(S)$, $\|v_r - g\|_\infty \rightarrow 0$ as $r \rightarrow 1$. It follows that $\limsup_{r \rightarrow 1} \|u_r - f\|_p \leq 2\varepsilon$. Since ε is arbitrary, $\|u_r - f\|_p \rightarrow 0$, as desired. $\square$

Theorem 6.6 fails when $p = \infty$. In fact, for $f \in L^\infty(S)$ and $u = P[f]$, we have $\|u_r - f\|_\infty \rightarrow 0$ as $r \rightarrow 1$ if and only if $f \in C(S)$, as the reader should verify.

In the case $\mu \in M(S)$ and $u = P[\mu]$, one might ask if the L^1 -functions u_r always converge to μ in $M(S)$. Here as well the answer is negative. Because $L^1(S)$ is a closed subspace of $M(S)$, $u_r \rightarrow \mu$ in $M(S)$ precisely when μ is absolutely continuous with respect to σ .

Nevertheless, we will see in the next section that there is a weak sense in which convergence at the boundary occurs in these two cases.

Weak* Convergence

A very useful concept in analysis is the notion of weak* convergence. Suppose X is a normed linear space and X^* is the dual space of X . If $\Lambda_1, \Lambda_2, \dots \in X^*$, then the sequence (Λ_k) is said to converge weak* to $\Lambda \in X^*$ provided $\lim_{k \rightarrow \infty} \Lambda_k(x) = \Lambda(x)$ for every $x \in X$. In other words, $\Lambda_k \rightarrow \Lambda$ weak* precisely when the sequence (Λ_k) converges pointwise on X to Λ . We will also deal with one-parameter families $\{\Lambda_r : r \in [0, 1)\} \subset X^*$; here we say $\Lambda_r \rightarrow \Lambda$ weak* provided $\Lambda_r(x) \rightarrow \Lambda(x)$ as $r \rightarrow 1$ for each fixed $x \in X$.

A simple observation that we need later is the following: If $\Lambda_k \rightarrow \Lambda$ weak*, then

$$6.7 \quad \|\Lambda\| \leq \liminf_{k \rightarrow \infty} \|\Lambda_k\|.$$

Here $\|\Lambda\|$ is the usual operator norm on X^* defined by $\|\Lambda\| = \sup\{|\Lambda(x)| : x \in X, \|x\| \leq 1\}$.

Convergence in norm implies weak* convergence, but the converse is false. A simple example is furnished by ℓ^2 , the space of square summable sequences. Because ℓ^2 is a Hilbert space, $(\ell^2)^* = \ell^2$. Let e_k denote the element of ℓ^2 that has 1 in the k^{th} slot and 0's elsewhere. Then for each $a = (a_1, a_2, \dots) \in \ell^2$, $\langle a, e_k \rangle = a_k$ for every k . Thus $e_k \rightarrow 0$ weak* in ℓ^2 as $k \rightarrow \infty$, while $\|e_k\| = 1$ in the ℓ^2 -norm for every k . This example also shows that inequality may occur in 6.7.

Our next result is the replacement for Theorem 6.6 in the cases mentioned at the end of the last section.

6.8 Theorem: *Poisson integrals have the following weak* convergence properties:*

- (a) If $\mu \in M(S)$ and $u = P[\mu]$, then $u_r \rightarrow \mu$ weak* in $M(S)$ as $r \rightarrow 1$.
- (b) If $f \in L^\infty(S)$ and $u = P[f]$, then $u_r \rightarrow f$ weak* in $L^\infty(S)$ as $r \rightarrow 1$.

PROOF: Recall that $C(S)^* = M(S)$. Suppose $\mu \in M(S)$, $u = P[\mu]$, and $g \in C(S)$. To prove (a), we need to show

$$6.9 \quad \int_S g u_r d\sigma \rightarrow \int_S g d\mu$$

as $r \rightarrow 1$.

Working with the left-hand side of 6.9, we have

$$\begin{aligned} \int_S g u_r d\sigma &= \int_S g(\zeta) \int_S P(r\zeta, \eta) d\mu(\eta) d\sigma(\zeta) \\ &= \int_S \int_S P(r\eta, \zeta) g(\zeta) d\sigma(\zeta) d\mu(\eta) \\ &= \int_S P[g](r\eta) d\mu(\eta), \end{aligned}$$

where we have used 6.5 again. Because $g \in C(S)$, $P[g](r\eta) \rightarrow g(\eta)$ uniformly on S as $r \rightarrow 1$. This proves 6.9 and completes the proof of (a).

The proof of (b) is similar. We first recall that $L^1(S)^* = L^\infty(S)$. With $f \in L^\infty(S)$ and $u = P[f]$, we thus need to show that

$$6.10 \quad \int_S g u_r d\sigma \rightarrow \int_S g f d\sigma$$

as $r \rightarrow 1$, for each $g \in L^1(S)$. Using the same manipulations as above, the left side of 6.10 equals $\int_S (P[g])_r f d\sigma$. By Theorem 6.6, $(P[g])_r \rightarrow g$ in $L^1(S)$ as $r \rightarrow 1$. Because $f \in L^\infty(S)$, we therefore have $(P[g])_r f \rightarrow g f$ in $L^1(S)$. This proves 6.10, and completes the proof of (b). $\square$

In Chapter 2 we told the reader that every bounded harmonic function on B is the Poisson integral of a bounded measurable function on S . In Chapter 3, we claimed that each positive harmonic function on B is the Poisson integral of some positive measure on S . We will prove these results in the next section. The key to these proofs is the following fundamental theorem on weak* convergence.

6.11 Theorem: *If X is a separable normed linear space, then every norm-bounded sequence in X^* contains a weak* convergent subsequence.*

PROOF: Assume (Λ_m) is a norm-bounded sequence in X^* . Then (Λ_m) is both pointwise bounded and equicontinuous on X . (Equicontinuity follows from the linearity of the functionals Λ_m .) By the Arzela-Ascoli Theorem for separable metric spaces (Theorem 11.28 in [9]), (Λ_m) contains a subsequence (Λ_{m_k}) converging

uniformly on compact subsets of X . In particular, (Λ_{m_k}) converges pointwise on X , which implies (Λ_{m_k}) converges weak* to some element of X^* . $\square$

In the next section we will apply the preceding theorem to the separable Banach spaces $C(S)$ and $L^q(S)$, $1 \leq q < \infty$.

The Spaces $h^p(B)$

The estimates obtained in Theorem 6.4 suggest the definition of some new function spaces. For $1 \leq p \leq \infty$, we define $h^p(B)$ to be the class of functions u harmonic on B for which

$$\|u\|_{h^p} = \sup_{0 \leq r < 1} \|u_r\|_p < \infty.$$

Thus $h^p(B)$ consists of the harmonic functions on B whose L^p -norms on spheres centered at the origin are uniformly bounded. Note that $h^\infty(B)$ is simply the collection of bounded harmonic functions on B , and that

$$\|u\|_{h^\infty} = \sup_{x \in B} |u(x)|.$$

We refer to the spaces $h^p(B)$ as “harmonic Hardy spaces”. The usual “Hardy spaces”, denoted by $H^p(B_2)$, consist of the functions in $h^p(B_2)$ that are holomorphic in B_2 ; they are named in honor of the mathematician G. H. Hardy, who first studied them.

It is straightforward to verify that $h^p(B)$ is a normed linear space under the norm $\|\cdot\|_{h^p}$. (A consequence of Theorem 6.12 below is that $h^p(B)$ is a Banach space.)

Here are some observations that can be elegantly stated in terms of the h^p -spaces:

- (a) The map $\mu \rightarrow P[\mu]$ is a linear isometry of $M(S)$ into $h^1(B)$.
- (b) For $1 < p \leq \infty$, the map $f \rightarrow P[f]$ is a linear isometry of $L^p(S)$ into $h^p(B)$.

Let us verify these claims. First, the maps in question are clearly linear. Second, in the case of (a), we have

$$\|P[\mu]\|_{h^1} \leq \|\mu\|$$

by Theorem 6.4(a). On the other hand, 6.7 and Theorem 6.8(a) show

$$\|\mu\| \leq \liminf_{r \rightarrow 1} \|(P[\mu])_r\|_1 \leq \|P[\mu]\|_{h^1}.$$

This proves (a). The proof of (b) is similar: Theorem 6.4(b) implies $\|P[f]\|_{h^p} \leq \|f\|_p$ for $1 < p \leq \infty$ and $f \in L^p(S)$. For $1 < p < \infty$, Theorem 6.6 yields $\|f\|_p \leq \|P[f]\|_{h^p}$, while $\|f\|_\infty \leq \|P[f]\|_{h^\infty}$ is implied by 6.7 and Theorem 6.8(b).

We now prove the remarkable result that the maps in (a) and (b) above are onto.

6.12 Theorem: *The Poisson integral induces the following surjective isometries:*

- (a) *The map $\mu \rightarrow P[\mu]$ is a linear isometry of $M(S)$ onto $h^1(B)$.*
- (b) *For $1 < p \leq \infty$, the map $f \rightarrow P[f]$ is a linear isometry of $L^p(S)$ onto $h^p(B)$.*

PROOF: All that remains to be verified in (a) is that the range of the map $\mu \mapsto P[\mu]$ is all of $h^1(B)$. To prove this, suppose $u \in h^1(B)$. By definition, this means that the family $\{u_r : r \in [0, 1]\}$ is norm-bounded in $L^1(S)$, and hence in $M(S) = C(S)^*$. Theorem 6.11 thus implies there exists a sequence $r_j \rightarrow 1$ such that the sequence u_{r_j} converges weak* to some $\mu \in M(S)$. The proof of (a) will be completed by showing that $u = P[\mu]$.

Fix $x \in B$. Because the functions $y \mapsto u(r_j y)$ are harmonic on $\bar{B}$, we have

$$6.13 \quad u(r_j x) = \int_S P(x, \zeta) u(r_j \zeta) d\sigma(\zeta)$$

for each j . Now let $j \rightarrow \infty$. Simply by continuity, the left side of 6.13 converges to $u(x)$. On the other hand, because $P(x, \cdot) \in C(S)$, the right side of 6.13 converges to $P[\mu](x)$. Therefore $u(x) = P[\mu](x)$, and thus $u = P[\mu]$ in B , as desired.

The proof of (b) is similar. Fix $p \in (1, \infty]$, let $u \in h^p(B)$, and let q be the index conjugate to p . Then the family $\{u_r : r \in [0, 1]\}$ is norm-bounded in $L^p(S) = L^q(S)^*$. By Theorem 6.11, there exists a sequence $r_j \rightarrow 1$ such that u_{r_j} converges weak* to some $f \in L^p(S)$. The argument given in the second paragraph may now be used, essentially verbatim, to show that $u = P[f]$; the difference is that

here we need to observe that $P(x, \cdot) \in L^q(S)$. We leave it to the reader to fill in the rest of the proof. $\square$

The theorem above contains the assertion made in Chapter 2, namely, that if u is bounded and harmonic in B , then $u = P[f]$ for some $f \in L^\infty(S)$. We next take up the claim made in Chapter 3.

6.14 Corollary: *If u is positive and harmonic in B , then there is a unique positive measure $\mu \in M(S)$ such that $u = P[\mu]$.*

PROOF: Suppose u is positive and harmonic in B . Then

$$\int_S |u_r| d\sigma = \int_S u_r d\sigma = u(0)$$

for any $r \in [0, 1)$, the last equality following from the mean-value property. Thus $u \in h^1(B)$, which by Theorem 6.12 means that there is a unique $\mu \in M(S)$ such that $u = P[\mu]$. Being the weak* limit of the positive measures u_r (Theorem 6.8(a)), μ is itself positive. $\square$

We conclude this section with a result that will be useful in the next chapter.

6.15 Theorem: *Let $\zeta \in S$. Suppose that u is positive and harmonic on B , and that u extends continuously to $\bar{B} \setminus \{\zeta\}$ with $u = 0$ on $S \setminus \{\zeta\}$. Then there exists a positive constant c such that*

$$u = cP(\cdot, \zeta).$$

PROOF: We have $u = P[\mu]$ for some positive $\mu \in M(S)$ by Theorem 6.14, and we have $u_r \rightarrow \mu$ weak* in $M(S)$ as $r \rightarrow 1$ by Theorem 6.8. The hypotheses on u imply that the functions u_r converge to 0 uniformly on compact subsets of $S \setminus \{\zeta\}$ as $r \rightarrow 1$. Therefore $\int_S \varphi d\mu = 0$ for any continuous φ on S that is zero near ζ . This implies that $\mu(S \setminus \{\zeta\}) = 0$, and thus that μ is a point mass at ζ . The conclusion of the theorem is immediate from this last statement. $\square$

The Schwarz Lemma

The Schwarz Lemma in complex analysis states that if h is holomorphic on B_2 with $|h| < 1$ and $h(0) = 0$, then $|h(z)| \leq |z|$ for all

$z \in B_2$; furthermore, if equality holds at any nonzero $z \in B_2$, then $h(z) = \lambda z$ for all $z \in B_2$, where λ is a complex number of modulus one. In this section we take up the Schwarz Lemma for functions harmonic on B_n .

Let S^+ denote the northern hemisphere $\{\zeta \in S : \zeta_n > 0\}$ and let S^- denote the southern hemisphere $\{\zeta \in S : \zeta_n < 0\}$. We define a harmonic function $U = U_n$ on B by setting

$$U = P[\chi_{S^+} - \chi_{S^-}].$$

In other words, U is the Poisson integral of the function that equals 1 on S^+ and -1 on S^- . Note that U is harmonic on B , with $|U| < 1$ and $U(0) = 0$.

The following theorem shows that U and its rotations are the extremal functions for the Schwarz Lemma for harmonic functions. Recall that $\mathbf{N} = (0, \dots, 0, 1)$ denotes the north pole of S .

6.16 Theorem: *Suppose u is harmonic on B , $|u| < 1$ on B , and $u(0) = 0$. Then*

$$|u(x)| \leq U(|x|\mathbf{N})$$

for every $x \in B$. Equality holds for some nonzero $x \in B$ if and only if $u = \lambda(U \circ T)$, where λ is a complex constant of modulus 1 and T is an orthogonal transformation.

PROOF: Fix $x \in B$. After a rotation, we can assume that x lies on the radius from 0 to $\mathbf{N}$, so that $x = |x|\mathbf{N}$.

First we consider the case where u is real valued. By Theorem 6.12, there is a real-valued function $f \in L^\infty(S)$ such that $u = P[f]$ and $\|f\|_\infty \leq 1$.

We claim that $u(x) \leq U(x)$. This inequality is equivalent to the inequality

$$\int_{S^-} P(x, \zeta)(1 + f(\zeta)) d\sigma \leq \int_{S^+} P(x, \zeta)(1 - f(\zeta)) d\sigma.$$

Because $x = |x|\mathbf{N}$, we have $P(x, \zeta) = (1 - |x|^2)/(1 + |x|^2 - 2|x|\zeta_n)^{n/2}$, so the inequality above is equivalent to

$$\begin{aligned}
6.17 \quad \int_{S^-} \frac{1+f(\zeta)}{(1+|x|^2-2|x|\zeta_n)^{n/2}} d\sigma(\zeta) \\
\leq \int_{S^+} \frac{1-f(\zeta)}{(1+|x|^2-2|x|\zeta_n)^{n/2}} d\sigma(\zeta).
\end{aligned}$$

The condition $u(0) = 0$ implies that $\int_{S^-} f d\sigma = -\int_{S^+} f d\sigma$. Thus, since ζ_n is negative on S^- and positive on S^+ , we have

$$\begin{aligned}
\int_{S^-} \frac{1+f(\zeta)}{(1+|x|^2-2|x|\zeta_n)^{n/2}} d\sigma(\zeta) &\leq \int_{S^-} \frac{1+f(\zeta)}{(1+|x|^2)^{n/2}} d\sigma(\zeta) \\
&= \int_{S^+} \frac{1-f(\zeta)}{(1+|x|^2)^{n/2}} d\sigma(\zeta) \\
&\leq \int_{S^+} \frac{1-f(\zeta)}{(1+|x|^2-2|x|\zeta_n)^{n/2}} d\sigma(\zeta).
\end{aligned}$$

Thus 6.17 holds, completing the proof that $u(x) \leq U(x)$. Note that if $x \neq 0$, then the last two inequalities are equalities if and only if $f = 1$ almost everywhere on S^+ and $f = -1$ almost everywhere on S^- . In other words, we have $u(x) = U(x)$ if and only if $u = U$.

Now remove the restriction that u be real valued. Choose $\beta \in \mathbf{C}$ such that $|\beta| = 1$ and $\beta u(x) = |u(x)|$. Apply the result just proved to the real part of βu , getting $|u(x)| \leq U(x)$, with equality if and only if $\beta u = U$. $\square$

Note that while the extremal functions for the holomorphic Schwarz Lemma are the entire functions $z \mapsto \lambda z$ (with $|\lambda| = 1$), the extremal functions for the harmonic Schwarz Lemma are discontinuous at the boundary of B . Later in this section we will give a concrete formula for U when $n = 2$; Exercise 21 of this chapter gives formulas for $U(|x|\mathbf{N})$ when $n = 3, 4$. The software package described in Appendix B can compute $U(|x|\mathbf{N})$ for higher values of n .

The Schwarz Lemma for holomorphic functions has a second part that we did not mention earlier: If h is holomorphic on B_2 and $|h| < 1$ on B_2 , then $|h'(0)| \leq 1$; equality holds if and only if $h(z) = \lambda z$ for some constant λ of modulus one. (Almost all complex analysis texts add the hypothesis that $h(0) = 0$, which is not needed for this part of the Schwarz Lemma.) The next theorem gives the corresponding result for harmonic functions. Here the gradient takes the place of the holomorphic derivative.

6.18 Theorem: Suppose u is a real-valued harmonic function on B_n and $|u| < 1$ on B_n . Then

$$|(\nabla u)(0)| \leq \frac{2V(B_{n-1})}{V(B_n)}.$$

Equality holds if and only if $u = U \circ T$ for some orthogonal transformation T .

PROOF: We begin by investigating the size of the partial derivative $D_n u(0)$. By Theorem 6.12, there is a real-valued function $f \in L^\infty(S)$ such that $u = P[f]$ and $\|f\|_\infty \leq 1$. Differentiating under the integral sign in the Poisson integral formula, we have

$$\begin{aligned} D_n u(0) &= \int_S D_n P(0, \zeta) f(\zeta) d\sigma(\zeta) \\ &= n \int_S \zeta_n f(\zeta) d\sigma(\zeta) \\ &\leq n \int_S |\zeta_n| d\sigma(\zeta). \end{aligned}$$

Equality holds here if and only if $f = 1$ almost everywhere on S^+ and $f = -1$ almost everywhere on S^- , which is equivalent to saying that u equals U . The last integral can be easily evaluated using A.7 from Appendix A:

$$\begin{aligned} n \int_S |\zeta_n| d\sigma(\zeta) &= \frac{2}{V(B_n)} \int_{B_{n-1}} (1 - |x|^2)^{-1/2} (1 - |x|^2)^{1/2} dV_{n-1}(x) \\ &= \frac{2V(B_{n-1})}{V(B_n)}. \end{aligned}$$

Thus $D_n u(0) \leq 2V(B_{n-1})/V(B_n)$, with equality if and only if $u = U$.

Applying this result to rotations of u , we see that every directional derivative of u at 0 is bounded above by $2V(B_{n-1})/V(B_n)$; the length of $\nabla u(0)$ is therefore bounded by the same constant, with equality if and only if u is a rotation of U . $\square$

The bound given above on $|(\nabla u)(0)|$ could not be improved if we added the hypothesis that $u(0) = 0$, because the extremal function already satisfies that condition.

When $n = 2$, the conclusion of the preceding theorem is that $|(\nabla u)(0)| \leq 4/\pi$. Note that the optimal constant $4/\pi$ is larger than 1, which is the optimal constant for the holomorphic Schwarz Lemma.

Theorem 6.18 fails for complex-valued harmonic functions (Exercise 20 of this chapter). The gradient, which points in the direction of maximal increase for a real-valued function, seems to have no natural geometric interpretation for complex-valued functions.

We now derive an explicit formula for U when $n = 2$. Here the arctangent of a real number is always taken to lie in the interval $(-\pi/2, \pi/2)$.

6.19 Proposition: *Let $z = (x, y)$ be a point in B_2 . Then*

$$U_2(x, y) = \frac{2}{\pi} \arctan \frac{2y}{1 - x^2 - y^2}$$

and

$$U_2(|z|N) = \frac{4}{\pi} \arctan |z|.$$

PROOF: Think of $z = x + iy$ as a complex variable. The conformal map $z \mapsto (1+z)/(1-z)$ takes B_2 onto the right half-plane. The function $z \mapsto \log[(1+z)/(1-z)]$, where $\log$ denotes the principal branch of the logarithm, is therefore holomorphic on B_2 . Multiplying the imaginary part of this function by $2/\pi$, we see that the function u defined by

$$u(x, y) = \frac{2}{\pi} \arctan \frac{2y}{1 - x^2 - y^2}$$

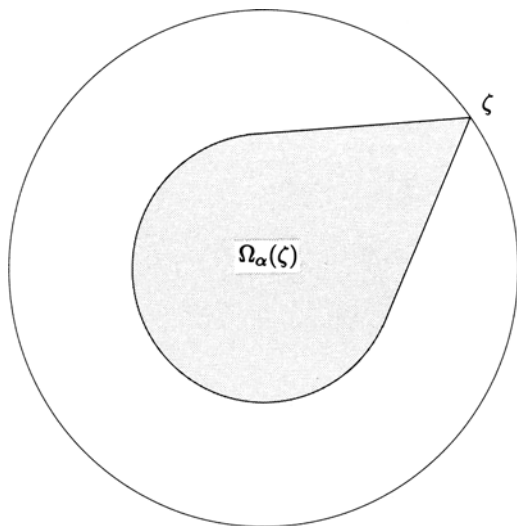
is harmonic on B_2 .

Because u is bounded on B_2 , Theorem 6.12 implies that $u = P[f]$ on B_2 for some $f \in L^\infty(S)$. Theorem 6.8 shows that $u_r \rightarrow f$ weak* in $L^\infty(S)$ as $r \rightarrow 1$. But note that u extends to be continuous on $B_2 \cup S^+ \cup S^-$, with $u = 1$ on S^+ and $u = -1$ on S^- ; thus $u_r \rightarrow u|_S$ weak* in $L^\infty(S)$ as $r \rightarrow 1$ (by the dominated convergence theorem). Hence $f = u|_S$ almost everywhere on S . Thus $u = U_2$, completing the proof of the first part of the proposition.

The second assertion in the proposition now follows from standard double-angle identities from trigonometry. $\square$

The Fatou Theorem

Recall the cones $\Gamma_\alpha(a)$ defined in the section *Limits Along Rays* of Chapter 2. We will use these cones to define nontangential approach regions in the ball. For $\alpha > 0$ and $\zeta \in S$, we first translate and rotate $\Gamma_\alpha(0)$ to obtain a new cone with vertex ζ and axis of symmetry containing the origin. This new cone crashes through the sphere on the side opposite of ζ , making it unsuitable for a nontangential approach region in B . To fix this, consider the largest ball $B(0, r(\alpha))$ contained in the new cone (we don't need to know the exact value of $r(\alpha)$). Taking the convex hull of $B(0, r(\alpha))$ and the point ζ , and then removing the point ζ , we obtain the open set $\Omega_\alpha(\zeta)$ pictured below. The region $\Omega_\alpha(\zeta)$ has the properties we seek for a nontangential approach region in B with vertex ζ : It stays away from the sphere except near ζ ; near ζ , it equals the translated and rotated version of $\Gamma_\alpha(0)$ with which we started.



The nontangential approach region $\Omega_\alpha(\zeta)$.

We have $\Omega_\alpha(\zeta) \subset \Omega_\beta(\zeta)$ if $\alpha < \beta$, and B is the union of the sets $\Omega_\alpha(\zeta)$ as α ranges over $(0, \infty)$.

Note that if T is an orthogonal transformation on $\mathbf{R}^n$, then $T(\Omega_\alpha(\zeta)) = \Omega_\alpha(T(\zeta))$. This allows us to transfer statements about the geometry of, say, $\Omega_\alpha(N)$ to any $\Omega_\alpha(\zeta)$.

A function u on B is said to have *nontangential limit* L at $\zeta \in S$ if, for each $\alpha > 0$, $u(x) \rightarrow L$ as $x \rightarrow \zeta$ within $\Omega_\alpha(\zeta)$.

In this section we prove that if $u \in h^1(B)$, then u has a nontangential limit at almost every $\zeta \in S$. (In this chapter, the term “almost everywhere” will mean “almost everywhere with respect to σ ”.) Theorems asserting the almost everywhere existence of limits within approach regions are commonly referred to as “Fatou Theorems”. The first such result was proved by Fatou [3], who in 1906 showed that bounded harmonic functions in the open unit disk have nontangential limits almost everywhere on the unit circle.

We approach the Fatou theorem for $h^1(B)$ via several operators known as “maximal functions”. Given a function u on B and $\alpha > 0$, the *nontangential maximal function* of u , denoted by $\mathcal{N}_\alpha[u]$, is the function on S defined by

$$\mathcal{N}_\alpha[u](\zeta) = \sup_{x \in \Omega_\alpha(\zeta)} |u(x)|.$$

The *radial maximal function* of u , denoted by $\mathcal{R}[u]$, is the function on S defined by

$$\mathcal{R}[u](\zeta) = \sup_{0 \leq r < 1} |u(r\zeta)|.$$

Clearly $\mathcal{R}[u](\zeta) \leq \mathcal{N}_\alpha[u](\zeta)$ for any $\zeta \in S$ and any $\alpha > 0$. The following theorem shows that, up to a constant multiple, the reverse inequality holds for positive harmonic functions in B .

6.20 Theorem: *For every $\alpha > 0$, there exists a constant $C_\alpha < \infty$ such that*

$$\mathcal{N}_\alpha[u](\zeta) \leq C_\alpha \mathcal{R}[u](\zeta)$$

for all $\zeta \in S$ and all positive harmonic functions u on B .

PROOF: Let $\zeta \in S$. The theorem then follows immediately from the existence of a constant C_α such that

$$6.21 \quad P(x, \eta) \leq C_\alpha P(|x|\zeta, \eta)$$

for all $x \in \Omega_\alpha(\zeta)$ and all $\eta \in S$. To prove 6.21, apply the law of cosines to the triangle with vertices 0, x , and ζ in 6.22 to see that there is a constant A_α such that

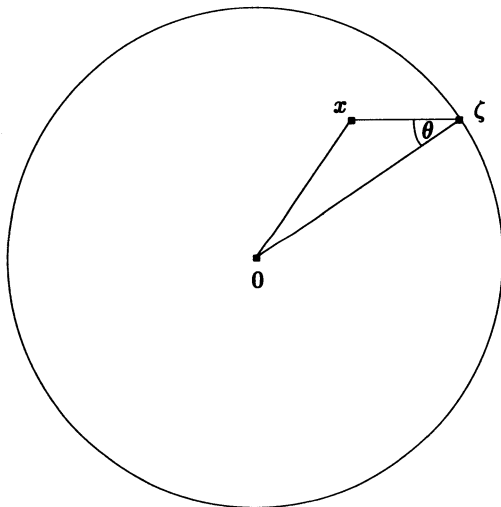
$$|x - \zeta| < A_\alpha(1 - |x|)$$

for all $x \in \Omega_\alpha(\zeta)$.

Thus

$$\begin{aligned} ||x|\zeta - \eta| &\leq ||x|\zeta - x| + |x - \eta| \\ &\leq |\zeta - x| + |x - \eta| \\ &\leq (A_\alpha + 1)|x - \eta| \end{aligned}$$

for all $x \in \Omega_\alpha(\zeta)$ and all $\eta \in S$. This shows that 6.21 holds with $C_\alpha = (1 + A_\alpha)^n$. $\square$



6.22 $|x - \zeta|$ is comparable to $(1 - |x|)$ for $x \in \Omega_\alpha(\zeta)$.

We turn now to an important operator in analysis, the Hardy-Littlewood maximal function. For $\zeta \in S$ and $\delta > 0$, define

$$\kappa(\zeta, \delta) = \{\eta \in S : |\eta - \zeta| < \delta\};$$

$\kappa(\zeta, \delta)$ is the open “spherical cap” on S with center ζ and radius δ . (Note that $\kappa(\zeta, \delta) = S$ when $\delta > 2$.) The *Hardy-Littlewood maximal function* of $\mu \in M(S)$, denoted by $\mathcal{M}[\mu]$, is the function on S defined by

$$\mathcal{M}[\mu](\zeta) = \sup_{\delta > 0} \frac{|\mu|(\kappa(\zeta, \delta))}{\sigma(\kappa(\zeta, \delta))}.$$

Suppose $\mu \in M(S)$ is positive and $\delta > 0$ is fixed. Let (ζ_j) be a sequence in S such that $\zeta_j \rightarrow \zeta$. Because the characteristic functions of $\kappa(\zeta_j, \delta)$ converge to 1 pointwise on $\kappa(\zeta, \delta)$ as $j \rightarrow \infty$, Fatou's Lemma shows that

$$\mu(\kappa(\zeta, \delta)) \leq \liminf_{j \rightarrow \infty} \mu(\kappa(\zeta_j, \delta)).$$

In other words, the function $\zeta \mapsto \mu(\kappa(\zeta, \delta))$ is lower-semicontinuous on S . From the definition of $\mathcal{M}[\mu]$, we conclude that $\mathcal{M}[\mu]$ is the supremum of lower-semicontinuous functions on S , and thus $\mathcal{M}[\mu]$ is lower-semicontinuous. In particular, $\mathcal{M}[\mu]: S \rightarrow [0, \infty]$ is Borel measurable.

In the next theorem we begin to see the connection between the Hardy-Littlewood maximal function and the Fatou Theorem.

6.23 Theorem: *If $\mu \in M(S)$ and $u = P[\mu]$, then*

$$\mathcal{R}[u](\zeta) \leq \mathcal{M}[\mu](\zeta)$$

for all $\zeta \in S$.

PROOF: We start with the following observation: If f is continuous, positive, and increasing on $[-1, 1]$, then given $\varepsilon > 0$, there exists a step function

$$\varphi = c_0 \chi_{[-1, 1]} + \sum_{j=1}^m c_j \chi_{(t_j, 1]}$$

such that $f \leq \varphi \leq f + \varepsilon$ on $[-1, 1]$; here $-1 < t_1 < \dots < t_m < 1$ and $c_0, \dots, c_m \in [0, \infty)$.

Turning to the proof of Theorem 6.23 proper, we may assume μ is positive and that $\zeta = N$. Fix $r \in [0, 1)$. Then $P(rN, \eta) = f(\eta_n)$, where

$$f(t) = \frac{1 - r^2}{(1 - 2rt + r^2)^{n/2}}$$

for $t \in [-1, 1]$. Let $\varepsilon > 0$. Because f has the properties specified in the first paragraph, there exists a step function φ as above with

$$P(r\mathbf{N}, \eta) \leq \varphi(\eta_n) \leq P(r\mathbf{N}, \eta) + \varepsilon$$

for all $\eta \in S$. Now for any $t \in \mathbf{R}$, the function on S defined by $\eta \mapsto \chi_{(t,1]}(\eta_n)$ is the characteristic function of an open cap centered at $\mathbf{N}$. We conclude that there are caps $\kappa_0, \dots, \kappa_m$, centered at $\mathbf{N}$, and nonnegative numbers $c_0, \dots, c_m$, such that

$$6.24 \quad P(r\mathbf{N}, \eta) \leq \sum_{j=0}^m c_j \chi_{\kappa_j}(\eta) \leq P(r\mathbf{N}, \eta) + \varepsilon$$

for all $\eta \in S$.

Integrating over S with respect to μ , 6.24 implies

$$\begin{aligned} u(r\mathbf{N}) &\leq \sum_{j=0}^m c_j \mu(\kappa_j) \\ &= \sum_{j=0}^m c_j \sigma(\kappa_j) (\mu(\kappa_j) / \sigma(\kappa_j)) \\ &\leq \mathcal{M}[\mu](\mathbf{N}) \left(\sum_{j=0}^m c_j \sigma(\kappa_j) \right) \\ &\leq \mathcal{M}[\mu](\mathbf{N}) (1 + \varepsilon). \end{aligned}$$

The last inequality follows by integrating 6.24 over S with respect to σ . Because ε is arbitrary, we have $u(r\mathbf{N}) \leq \mathcal{M}[\mu](\mathbf{N})$, and thus $\mathcal{R}[u](\mathbf{N}) \leq \mathcal{M}[\mu](\mathbf{N})$, as desired. $\square$

Theorem 6.29 below estimates the σ -measure of the set where $\mathcal{M}[\mu]$ is large. The “covering lemma” that we prove next will be a crucial ingredient in its proof. We abuse notation slightly and adopt the following convention: If $\kappa = \kappa(\zeta, \delta)$, then 3κ denotes the cap $\kappa(\zeta, 3\delta)$.

6.25 Covering Lemma: *Given caps $\kappa_j = \kappa(\zeta_j, \delta_j)$, $j = 1, \dots, m$, there exists a subset $J \subset \{1, \dots, m\}$ such that:*

(a) *The collection $\{\kappa_j : j \in J\}$ is pairwise disjoint;*

(b)
$$\bigcup_{j=1}^m \kappa_j \subset \bigcup_{j \in J} 3\kappa_j.$$

PROOF: We describe an inductive procedure for selecting the desired subcollection. Start by choosing a cap κ_{j_1} having the largest radius among the caps $\kappa_1, \dots, \kappa_m$. If all caps intersect κ_{j_1} , we stop. Otherwise, remove the caps intersecting κ_{j_1} , and from those remaining, select one of largest radius and denote it by κ_{j_2} . If all the remaining caps intersect κ_{j_2} , we stop; otherwise we continue as above. This process gives us a finite subcollection $\{\kappa_j : j \in J\}$, where $J = \{j_1, j_2, \dots\}$.

The subcollection $\{\kappa_j : j \in J\}$ clearly satisfies (a).

Given $\kappa \in \{\kappa_1, \dots, \kappa_m\}$, let κ' denote the first cap in the sequence $\kappa_{j_1}, \kappa_{j_2}, \dots$ such that $\kappa \cap \kappa'$ is non-empty. The way in which the caps in $\{\kappa_j : j \in J\}$ were chosen shows that the radius of κ' is at least as large as that of κ . By the triangle inequality, $\kappa \subset 3\kappa'$, proving (b). $\square$

In proving the next theorem we will need the fact that there exist constants $a > 0, A < \infty$, depending only on the dimension n , such that

$$6.26 \quad a\delta^{n-1} \leq \sigma(\kappa(\zeta, \delta)) \leq A\delta^{n-1}$$

for all $\zeta \in S$ and all $\delta \in (0, 2]$. Intuitively, $\kappa(\zeta, \delta)$ looks like an $(n-1)$ -dimensional ball of radius δ for small $\delta > 0$, indicating that 6.26 is correct. One may verify 6.26 rigorously by using formula A.3 in Appendix A.

From 6.26 we see that

$$6.27 \quad \sigma(3\kappa) \leq 3^{n-1}(A/a)\sigma(\kappa)$$

for all caps $\kappa \subset S$.

To motivate our next result, consider the following inequality for L^1 -functions: If $f \in L^1(S)$ and $t > 0$, then

$$t\sigma(\{|f| > t\}) \leq \int_{\{|f| > t\}} |f| d\sigma \leq \|f\|_1,$$

giving

$$6.28 \quad \sigma(\{|f| > t\}) \leq \frac{\|f\|_1}{t}.$$

Here we have used the abbreviated notation $\{|f| > t\}$ to denote the set $\{\zeta \in S : |f(\zeta)| > t\}$. The next theorem states that for $\mu \in M(S)$, the Hardy-Littlewood maximal function $\mathcal{M}[\mu]$ is almost in $L^1(S)$, in the sense that it satisfies an inequality resembling 6.28.

6.29 Theorem: *For every $\mu \in M(S)$ and every $t \in (0, \infty)$,*

$$\sigma(\{\mathcal{M}[\mu] > t\}) \leq \frac{C\|\mu\|}{t},$$

where $C = 3^{n-1}(A/a)$.

PROOF: Suppose $t \in (0, \infty)$. Let $E \subset \{\mathcal{M}[\mu] > t\}$ be compact. Then for each $\zeta \in E$, there is a cap κ centered at ζ with $|\mu|(\kappa)/\sigma(\kappa) > t$. Being compact, E is covered by finitely many such caps. From these we may choose a subcollection with the properties specified in 6.25. Thus there are pairwise disjoint caps $\kappa_1, \dots, \kappa_N$ such that $3\kappa_1, \dots, 3\kappa_N$ cover E , and such that $|\mu|(\kappa_j)/\sigma(\kappa_j) > t$ for $j = 1, \dots, N$. By 6.27 and the definition of C , we therefore have

$$\sigma(E) \leq \sum_{j=1}^N \sigma(3\kappa_j) \leq C \sum_{j=1}^N \sigma(\kappa_j) \leq C \sum_{j=1}^N \frac{|\mu|(\kappa_j)}{t} \leq \frac{C\|\mu\|}{t};$$

the pairwise disjointness of the caps $\kappa_1, \dots, \kappa_N$ was used in the last inequality. Taking the supremum over all compact $E \subset \{\mathcal{M}[\mu] > t\}$ now gives the conclusion of the theorem. $\square$

Let us write $\mathcal{M}[f]$ in place of $\mathcal{M}[\mu]$ when $d\mu = f d\sigma$ for $f \in L^1(S)$. The conclusion of Theorem 6.29 for $f \in L^1(S)$ is then

$$\mathbf{6.30} \quad \sigma(\{\mathcal{M}[f] > t\}) \leq \frac{C\|f\|_1}{t}.$$

We now prove the Fatou theorem for Poisson integrals of L^1 -functions.

6.31 Theorem: *If $f \in L^1(S)$, then $P[f]$ has nontangential limit $f(\zeta)$ at almost every $\zeta \in S$.*

PROOF: For $f \in L^1(S)$ and $\alpha > 0$, define the function $\mathcal{L}_\alpha[f]$ on S by

$$\mathcal{L}_\alpha[f](\zeta) = \limsup_{\substack{x \rightarrow \zeta \\ x \in \Omega_\alpha(\zeta)}} |P[f](x) - f(\zeta)|.$$

We first show that $\mathcal{L}_\alpha[f] = 0$ almost everywhere on S .

Note that

$$6.32 \quad \mathcal{L}_\alpha[f] \leq \mathcal{N}_\alpha[P[|f|]] + |f|,$$

and that $\mathcal{L}_\alpha[f_1 + f_2] \leq \mathcal{L}_\alpha[f_1] + \mathcal{L}_\alpha[f_2]$ (both statements holding almost everywhere on S). Note also that $\mathcal{L}_\alpha[f] \equiv 0$ for every $f \in C(S)$.

Now fix $f \in L^1(S)$ and $\alpha > 0$. Also fixing $t \in (0, \infty)$, our main goal is to show that $\sigma(\{\mathcal{L}_\alpha[f] > 2t\}) = 0$.

Given $\varepsilon > 0$, we may choose $g \in C(S)$ such that $\|f - g\|_1 < \varepsilon$. We then have

$$\begin{aligned} \mathcal{L}_\alpha[f] &\leq \mathcal{L}_\alpha[f - g] + \mathcal{L}_\alpha[g] \\ &= \mathcal{L}_\alpha[f - g] \\ &\leq \mathcal{N}_\alpha[P[|f - g|]] + |f - g| \\ &\leq C_\alpha \mathcal{R}[P[|f - g|]] + |f - g| \\ &\leq C_\alpha \mathcal{M}[|f - g|] + |f - g|, \end{aligned}$$

this holding at almost every point of S . In this string of inequalities we have used 6.32, 6.20, and 6.23 in succession.

We thus have

$$6.33 \quad \{\mathcal{L}_\alpha[f] > 2t\} \subset \{C_\alpha \mathcal{M}[|f - g|] > t\} \cup \{|f - g| > t\}.$$

By 6.30 and 6.28, the σ -measure of the right side of 6.33 is less than or equal to

$$\frac{CC_\alpha \|f - g\|_1}{t} + \frac{\|f - g\|_1}{t}.$$

Recalling that $\|f - g\|_1 < \varepsilon$ and that ε is arbitrary, we have shown that the set $\{\mathcal{L}_\alpha[f] > 2t\}$ is contained in sets of arbitrarily small σ -measure, and therefore $\sigma(\{\mathcal{L}_\alpha[f] > 2t\}) = 0$.

Because this is true for every $t \in (0, \infty)$, we have proved $\mathcal{L}_\alpha[f] = 0$ almost everywhere on S .

To finish, let $f \in L^1(S)$, and define $E_m = \{\mathcal{L}_m[f] = 0\}$ for $m = 1, 2, \dots$. We have shown that E_m is a set of full measure on S for each m , and thus $\bigcap E_m$ is a set of full measure. At each $\zeta \in \bigcap E_m$, $P[f]$ has nontangential limit $f(\zeta)$, which is what we set out to prove. $\square$

Recall that $\mu \in M(S)$ is said to be *singular* with respect to σ , written $\mu \perp \sigma$, if there exists a Borel set $E \subset S$ such that $\sigma(E) = 0$ and $|\mu|(E) = \|\mu\|$. Recall also that each $\mu \in M(S)$ has a unique decomposition $d\mu = f d\sigma + d\mu_s$, where $f \in L^1(S)$ and $\mu_s \perp \sigma$; this is called the *Lebesgue decomposition* of μ with respect to σ .

The following result is the second half of the Fatou Theorem for $h^1(B)$.

6.34 Theorem: *If $\mu \perp \sigma$, then $P[\mu]$ has nontangential limit 0 almost everywhere on S .*

PROOF: Much of the proof is similar to that of Theorem 6.31, and so we will be brief about certain details.

It suffices to prove the theorem for positive measures, so suppose $\mu \in M(S)$ is positive and that $\mu \perp \sigma$. For $\alpha > 0$, define

$$\mathcal{L}_\alpha[\mu](\zeta) = \limsup_{\substack{x \rightarrow \zeta \\ x \in \Omega_\alpha(\zeta)}} P[\mu](x)$$

for $\zeta \in S$. Fixing $t \in (0, \infty)$, the proof will be completed by showing $\sigma(\{\mathcal{L}_\alpha[\mu] > 2t\}) = 0$.

Let $\varepsilon > 0$. Because $\mu \perp \sigma$, the regularity of μ implies the existence of a compact set $K \subset S$ such that $\sigma(K) = 0$ and $\mu(S \setminus K) < \varepsilon$. Writing $\mu = \mu_1 + \mu_2$, with $d\mu_1 = \chi_K d\mu$ and $d\mu_2 = \chi_{S \setminus K} d\mu$, observe that $\mathcal{L}_\alpha[\mu_1] = 0$ on $S \setminus K$ (Exercise 2 of this chapter) and that $\|\mu_2\| = \mu(S \setminus K) < \varepsilon$.

Because $\mathcal{L}_\alpha[\mu] \leq \mathcal{L}_\alpha[\mu_1] + \mathcal{L}_\alpha[\mu_2]$, we have

$$\begin{aligned} \mathbf{6.35} \quad \{\mathcal{L}_\alpha[\mu] > 2t\} &\subset \{\mathcal{L}_\alpha[\mu_1] > t\} \cup \{\mathcal{L}_\alpha[\mu_2] > t\} \\ &\subset K \cup \{C_\alpha \mathcal{M}[\mu_2] > t\}. \end{aligned}$$

(The inequality $\mathcal{L}_\alpha[\mu_2] \leq C_\alpha \mathcal{M}[\mu_2]$ is obtained as in the proof of Theorem 6.31.) Recalling that $\sigma(K) = 0$, we see by Theorem 6.29 that the left side of 6.35 is contained in a set of σ -measure at most $(CC_\alpha \|\mu_2\|)/t$, which is less than $(CC_\alpha \varepsilon)/t$. Since ε is arbitrary, we conclude that $\sigma(\{\mathcal{L}_\alpha[\mu] > 2t\}) = 0$, as desired. $\square$

Theorems 6.31 and 6.34 immediately give the following result.

6.36 Corollary: *Suppose $\mu \in M(S)$ and $d\mu = f d\sigma + d\mu_s$ is the Lebesgue decomposition of μ with respect to σ . Then $P[\mu]$ has nontangential limit $f(\zeta)$ at almost every $\zeta \in S$.*

If $u \in h^1(B)$, then $u = P[\mu]$ for some $\mu \in M(S)$ by Theorem 6.12. Corollary 6.36 thus implies that u has nontangential limits almost everywhere on S . Because $h^p(B) \subset h^1(B)$ for all $p \in [1, \infty]$, the same is true of any $u \in h^p(B)$.

Exercises

1. Show that if $f \in L^1(S)$ and $\zeta \in S$ is a point of continuity of f , then $P[f]$ extends continuously to $B \cup \{\zeta\}$.
2. Suppose $V \subset S$ is open, $\mu \in M(S)$, and $|\mu|(V) = 0$. Show that for every $\zeta \in V$, $P[\mu](x) \rightarrow 0$ as $x \rightarrow \zeta$ unrestrictedly in B .
3. Suppose that $\mu \in M(S)$ and $\zeta \in S$. Show that

$$(1 - r)^{n-1} P[\mu](r\zeta) \rightarrow 2\mu(\{\zeta\})$$

as $r \rightarrow 1$.

4. Suppose that X is a normed linear space and that (Λ_j) is a weak* convergent sequence in X^* .
 - (a) Give an example to show that the sequence (Λ_j) need not be norm-bounded.
 - (b) Prove that if X is a Banach space, then the sequence (Λ_j) must be norm-bounded. (*Hint*: uniform boundedness principle.)
5. It is easy to see that if $\mu_j \rightarrow \mu$ in $M(S)$, then $P[\mu_j] \rightarrow P[\mu]$ uniformly on compact subsets of B . Prove that the conclusion is still valid if we assume only that $\mu_j \rightarrow \mu$ weak* in $M(S)$.
6. Suppose that (μ_j) is a norm-bounded sequence in $M(S)$ such that $(P[\mu_j])$ converges pointwise on B . Prove that (μ_j) is weak* convergent in $M(S)$.
7. Prove directly (that is, without the help of Theorem 6.12) that $h^p(B)$ is a Banach space for every $p \in [1, \infty]$.
8. Prove that a real-valued function on B belongs to $h^1(B)$ if and only if it is the difference of two positive harmonic functions on B .

9. Let $\zeta \in S$. Show that $P(\cdot, \zeta) \in h^p(B)$ for $p = 1$ but not for any $p > 1$.
10. (a) Suppose $p \geq 1$ and $u \in h^p(B)$. Prove that the function $(1 - |x|)^{(n-1)/p}u(x)$ is bounded on B .
 (b) Suppose $p > 1$ and $u \in h^p(B)$. Prove that

$$(1 - |x|)^{(n-1)/p}u(x) \rightarrow 0$$

as $|x| \rightarrow 1$.

11. A family of functions $\mathcal{F} \subset L^1(S)$ is said to be *uniformly integrable* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f| d\sigma < \varepsilon$ whenever $f \in \mathcal{F}$ and $\sigma(E) < \delta$. Show that a harmonic function u in B is the Poisson integral of a function in $L^1(S)$ if and only if the family $\{u_r : r \in [0, 1)\}$ is uniformly integrable.
12. Prove that there exists $u \in h^1(B)$ such that $u(B \cap B(\zeta, \varepsilon)) = \mathbf{R}$ for all $\zeta \in S$, $\varepsilon > 0$. (*Hint*: Let $u = P[\mu]$, where μ is a judiciously chosen sum of point masses.)
13. Suppose that $p \in [1, \infty)$ and u is harmonic on B . Show that $u \in h^p(B)$ if and only if there exists a harmonic function v on B such that $|u|^p \leq v$ in B .
14. Suppose $n > 2$. Show that if u is positive and harmonic on $\{x \in \mathbf{R}^n : |x| > 1\}$, then there exists a unique positive measure $\mu \in M(S)$ and a unique nonnegative constant c such that

$$u(x) = P_E[\mu](x) + c(1 - |x|^{2-n})$$

for $|x| > 1$. (Here P_E is the external Poisson kernel defined in Chapter 4.) State and prove an analogous result for the case $n = 2$.

15. Let Ω denote B_3 minus the x_3 -axis. Show that every bounded harmonic function on Ω extends to be harmonic on B_3 .
16. Suppose $\zeta \in S$, and f is positive and continuous on $S \setminus \{\zeta\}$. Need there exist a positive harmonic function u on B that extends continuously to $\bar{B} \setminus \{\zeta\}$ with $u = f$ on $S \setminus \{\zeta\}$?

17. Suppose $\sigma(E) = 0$. Prove that there exists a positive harmonic function u on B such that u has nontangential limit ∞ at every point of E .
18. Let $\mathcal{F}$ denote the family of positive harmonic functions u on B such that $u(0) = 0$. Compute $\inf\{u(N/2) : u \in \mathcal{F}\}$ and $\sup\{u(N/2) : u \in \mathcal{F}\}$. Do there exist functions in $\mathcal{F}$ that attain either of these extreme values at $N/2$? If so, are they unique?
19. Find all extreme points of $\mathcal{F}$, where $\mathcal{F}$ is the family defined in the previous exercise. (A function in $\mathcal{F}$ is called an *extreme point* of $\mathcal{F}$ if it cannot be written as the average of two distinct functions in $\mathcal{F}$.)
20. Show that the bound on $|(\nabla u)(0)|$ given by Theorem 6.18 can fail if the requirement that u be real valued is dropped.
21. Show that

$$U_3(|x|N) = \frac{1}{|x|} \left[1 - \frac{1 - |x|^2}{\sqrt{1 + |x|^2}} \right]$$

and

$$U_4(|x|N) = \frac{2}{\pi} \frac{(1 + |x|^2)^2 \arctan |x| - |x|(1 - |x|^2)}{|x|^2(1 + |x|^2)}.$$

(*Hint:* Evaluate the Poisson integrals that define $U_3(|x|N)$ and $U_4(|x|N)$, using an appropriate result from Appendix A. Be prepared for some hard calculus.)

22. Suppose u is harmonic on B and $\sum_{m=0}^{\infty} p_m$ is the homogeneous expansion of u about 0. Prove that

$$\|u\|_{h^2} = \left(\sum_{m=0}^{\infty} \int_S |p_m|^2 d\sigma \right)^{1/2}.$$

23. *Schwarz Lemma for h^2 -functions:* Suppose u is harmonic on B and $\|u\|_{h^2} \leq 1$.

(a) Show that if $u(0) = 0$, then $|u(x)| \leq \sqrt{n}|x|$ for all $x \in B$.

(b) Show that $|\nabla u(0)| \leq \sqrt{n}$.

(c) Identify extremal functions in (a) and (b), thereby proving these estimates are sharp.

24. For a smooth function u on B , we define the *radial derivative* $D_R u$ by setting $D_R u(x) = \nabla u(x) \cdot x$ for all $x \in B$. Show that there exist positive constants c and C , depending only on the dimension n , such that

$$c\|u\|_{h^2} \leq |u(0)| + \left(\int_B |D_R u(x)|^2 (1 - |x|) dV(x) \right)^{1/2} \leq C\|u\|_{h^2}$$

for all u harmonic on B . (*Hint:* Use the homogeneous expansion of u , Exercise 20 in Chapter 1, and polar coordinates.)

25. (a) Find a measure $\mu \in M(S)$ with $\mathcal{M}[\mu] \notin L^1(S)$.
 (b) Can the measure μ in part (a) be chosen to be absolutely continuous with respect to σ ?
26. Let $\mu \in M(S)$. Show that if

$$\lim_{\delta \rightarrow 0} \frac{\mu(\kappa(\zeta, \delta))}{\sigma(\kappa(\zeta, \delta))} = L \in \mathbf{C},$$

then $\lim_{r \rightarrow 1} P[\mu](r\zeta) = L$. (*Suggestion:* Without loss of generality, $\zeta = \mathbf{N}$. For η near $\mathbf{N}$, approximate $P(r\mathbf{N}, \eta)$ as in the proof of Theorem 6.23.)

27. Let $f \in L^1(S)$. A point $\zeta \in S$ is called a *Lebesgue point* of f if

$$\lim_{\delta \rightarrow 0} \frac{1}{\sigma(\kappa(\zeta, \delta))} \int_{\kappa(\zeta, \delta)} |f - f(\zeta)| d\sigma = 0.$$

Show that almost every $\zeta \in S$ is a Lebesgue point of f . (*Hint:* Imitate the proof of Theorem 6.31.)

28. For u a function on B , let $u^*(\zeta)$ denote the nontangential limit of u at $\zeta \in S$, provided this limit exists. Show that if $1 < p \leq \infty$ and $u \in h^p(B)$, then $u = P[u^*]$, while this need not hold if $u \in h^1(B)$.

29. Consider the holomorphic function f on B_2 defined by $f(z) = e^{(1+z)/(1-z)}$. Show that f has a nontangential limit bounded by 1 at every point of S , even though f is unbounded in B_2 . Explain why this does not contradict h^p -theory.

CHAPTER 7

Harmonic Functions on Half-Spaces

In this chapter we study harmonic functions defined on the upper half-space H . Harmonic function theory on H has a distinctly different flavor from that on B . One advantage of H over B is the dilation-invariance of H . We have already put this to good use in the section *Limits Along Rays* in Chapter 2. Some disadvantages: ∂H is not compact and Lebesgue measure on ∂H is not finite.

Recall our notation: The upper half-space $H = H_n$ is the set

$$H = \{(x, y) \in \mathbf{R}^n : y > 0\}.$$

We identify $\mathbf{R}^n$ with $\mathbf{R}^{n-1} \times \mathbf{R}$, writing a typical point $z \in \mathbf{R}^n$ as $z = (x, y)$, where $x \in \mathbf{R}^{n-1}$ and $y \in \mathbf{R}$. We also identify $\mathbf{R}^{n-1}$ with $\mathbf{R}^{n-1} \times \{0\}$; with this convention we have $\partial H = \mathbf{R}^{n-1}$.

For u a function on H and $y > 0$, we let u_y denote the function on $\mathbf{R}^{n-1}$ defined by

$$u_y(x) = u(x, y).$$

The functions u_y play the same role on the upper half-space that the dilations play on the ball.

The Poisson Kernel for the Upper Half-Space

We seek a function P_H on $H \times \mathbf{R}^{n-1}$ analogous to the Poisson kernel for the ball. Thus, for each fixed $t \in \mathbf{R}^{n-1}$, we would like $P_H(\cdot, t)$ to be a positive harmonic function on H having the appropriate approximate-identity properties (see 1.16).

Fix $t = 0$ for the moment; we will concentrate first on finding $P_H(\cdot, 0)$. Taking our cue from Theorem 6.15, we look for a positive harmonic function on H that extends continuously to $\bar{H} \setminus \{0\}$ with boundary values 0 on $\mathbf{R}^{n-1} \setminus \{0\}$. One such function is $u(x, y) = y$, but this is obviously not what we want— u doesn't "blow up" at 0 as we know $P_H(\cdot, 0)$ should. On the other hand, u does blow up at ∞ . Applying the Kelvin transform, we can move the singularity of u from ∞ to 0 and arrive at the desired function.

Thus, with $u(x, y) = y$, let us define

$$v = K[u]$$

on H , where K is the Kelvin transform introduced in Chapter 4. A simple computation shows that

$$v(x, y) = \frac{y}{(|x|^2 + y^2)^{n/2}}$$

for all $(x, y) \in H$. Because the inversion map preserves the upper half-space and the Kelvin transform preserves harmonic functions, we know without any computation that v is a positive harmonic function on H .

The function v has the following important property: $v_y(x) = y^{-(n-1)}v_1(x/y)$ for all $(x, y) \in H$. The change of variables $x \mapsto yx'$ shows therefore that $\int_{\mathbf{R}^{n-1}} v_y(x) dx$ is the same for all $y > 0$. (Here dx denotes Lebesgue measure on $\mathbf{R}^{n-1}$.) Because $\int_{\mathbf{R}^{n-1}} v_1(x) dx < \infty$ (verify using polar coordinates), there exists a positive constant c_n such that

$$c_n \int_{\mathbf{R}^{n-1}} v_y(x) dx = c_n \int_{\mathbf{R}^{n-1}} \frac{y}{(|x|^2 + y^2)^{n/2}} dx = 1$$

for all $y > 0$. At the end of this section we will show that $c_n = 2/(nV(B_n))$.

The function $c_n v$ has all the properties we sought for $P_H(\cdot, 0)$. To obtain $P_H(\cdot, t)$, we simply translate $c_n v$ by t . Thus we now make our official definition: For $z = (x, y) \in H$ and $t \in \mathbf{R}^{n-1}$, set

$$P_H(z, t) = c_n \frac{y}{(|x - t|^2 + y^2)^{n/2}}.$$

The function P_H is called the *Poisson kernel* for the upper half-space.

Note that P_H can be written as

$$P_H(z, t) = c_n \frac{y}{|z - t|^n}.$$

In this form, P_H reminds us of the Poisson kernel for the ball. (If $(x, y) \in H$, then y is the distance from (x, y) to ∂H ; analogously, the numerator of the Poisson kernel for B is roughly the distance to ∂B .)

The work above shows that $P_H(\cdot, t)$ is positive and harmonic on H for each $t \in \mathbf{R}^{n-1}$, and that

$$7.1 \quad \int_{\mathbf{R}^{n-1}} P_H(z, t) dt = 1$$

for each $z \in H$. The next result gives the remaining approximate-identity property that we need for the Dirichlet problem on H .

7.2 Proposition: For every $a \in \mathbf{R}^{n-1}$ and every $\delta > 0$,

$$\int_{|t-a|>\delta} P_H(z, t) dt \rightarrow 0$$

as $z \rightarrow a$.

We leave the proof of Proposition 7.2 to the reader; it follows without difficulty from the definition of P_H .

Let us now evaluate the normalizing constant c_n . We accomplish this with a slightly underhanded trick:

$$\begin{aligned}
\frac{1}{c_n} &= \frac{2}{\pi} \int_0^\infty \frac{1}{1+y^2} \int_{\mathbf{R}^{n-1}} \frac{y}{(|x|^2 + y^2)^{n/2}} dx dy \\
&= \frac{2}{\pi} \int_H \frac{y}{(1+y^2)(|x|^2 + y^2)^{n/2}} dx dy \\
&= \frac{2nV(B)}{\pi} \int_{S^+} \int_0^\infty \frac{\zeta_n}{1+(r\zeta_n)^2} dr d\sigma(\zeta) \\
&= nV(B)/2,
\end{aligned}$$

where the third equality is obtained by switching to polar coordinates (S^+ denotes the upper half-sphere).

The Dirichlet Problem for the Upper Half-Space

For μ a complex Borel measure on $\mathbf{R}^{n-1}$, the Poisson integral of μ , denoted by $P_H[\mu]$, is the function on H defined by

$$P_H[\mu](z) = \int_{\mathbf{R}^{n-1}} P_H(z, t) d\mu(t).$$

We can verify that $P_H[\mu]$ is harmonic on H by differentiating under the integral sign, or by noting that $P_H[\mu]$ satisfies the volume version of the mean-value property on H .

The set of complex Borel measures on $\mathbf{R}^{n-1}$ will be denoted by $M(\mathbf{R}^{n-1})$. With the total variation norm $\|\cdot\|$, the Banach space $M(\mathbf{R}^{n-1})$ is the dual space of $C_0(\mathbf{R}^{n-1})$, the space of continuous functions f on $\mathbf{R}^{n-1}$ that vanish at ∞ (equipped with the supremum norm).

For $1 \leq p < \infty$, $L^p(\mathbf{R}^{n-1})$ denotes the space of Borel measurable functions on $\mathbf{R}^{n-1}$ for which

$$\|f\|_p = \left(\int_{\mathbf{R}^{n-1}} |f(x)|^p dx \right)^{1/p} < \infty;$$

$L^\infty(\mathbf{R}^{n-1})$ consists of the Borel measurable functions f on $\mathbf{R}^{n-1}$ for which $\|f\|_\infty < \infty$, where $\|f\|_\infty$ denotes the essential supremum norm on $\mathbf{R}^{n-1}$ with respect to Lebesgue measure.

Recall that on S , $L^p(S) \subset L^q(S)$ whenever $p > q$. On $\mathbf{R}^{n-1}$, neither of the spaces $L^p(\mathbf{R}^{n-1})$, $L^q(\mathbf{R}^{n-1})$ contains the other when

$p \neq q$. The reader should keep this in mind as we develop Poisson-integral theory in this new setting.

The Poisson integral of $f \in L^p(\mathbf{R}^{n-1})$, for any $p \in [1, \infty]$, is the function $P_H[f]$ on H defined by

$$P_H[f](z) = \int_{\mathbf{R}^{n-1}} P_H(z, t) f(t) dt.$$

Because $P_H(z, \cdot)$ belongs to $L^q(\mathbf{R}^{n-1})$ for every $q \in [1, \infty]$, the integral above is well-defined for any $z \in H$ (by Hölder's inequality). An argument like the one given for $P_H[\mu]$ shows that $P_H[f]$ is harmonic on H .

We now prove a result that the reader has surely already guessed.

7.3 Solution of the Dirichlet Problem for H : Suppose f is continuous and bounded on $\mathbf{R}^{n-1}$. Define u on $\bar{H}$ by

$$u(z) = \begin{cases} P_H[f](z) & \text{if } z \in H, \\ f(z) & \text{if } z \in \mathbf{R}^{n-1}. \end{cases}$$

Then u is continuous on $\bar{H}$ and harmonic on H . Moreover,

$$|u| \leq \|f\|_\infty$$

on $\bar{H}$.

PROOF: The estimate $|u| \leq \|f\|_\infty$ on $\bar{H}$ is immediate from 7.1. We already know that u is harmonic on H . The proof that u is continuous on $\bar{H}$ is like that of Theorem 1.14: Let $a \in \mathbf{R}^{n-1}$ and $\delta > 0$. Then

$$\begin{aligned} |u(z) - f(a)| &= \left| \int_{\mathbf{R}^{n-1}} P_H(z, t) (f(t) - f(a)) dt \right| \\ &\leq \int_{|t-a| \leq \delta} P_H(z, t) |f(t) - f(a)| dt \\ &\quad + 2\|f\|_\infty \int_{|t-a| > \delta} P_H(z, t) dt \end{aligned}$$

for all $z \in H$. If δ is small, the integral over $\{|t-a| \leq \delta\}$ will be small by the continuity of f at a and 7.1. The integral over $\{|t-a| > \delta\}$ approaches 0 as $z \rightarrow a$ by Proposition 7.2. $\square$

In the special case where f is uniformly continuous on $\mathbf{R}^{n-1}$, we can make a stronger assertion:

7.4 Theorem: *If f is bounded and uniformly continuous on $\mathbf{R}^{n-1}$ and $u = P_H[f]$, then $u_y \rightarrow f$ uniformly on $\mathbf{R}^{n-1}$ as $y \rightarrow 0$.*

PROOF: The uniform continuity of f on $\mathbf{R}^{n-1}$ shows that the estimates in the proof of Theorem 7.3 can be made uniformly in a . $\square$

See Exercise 4 of this chapter for a converse to Theorem 7.4.

The next result follows immediately from Corollary 2.2 and Theorem 7.3; we state it as a theorem because of its importance.

7.5 Theorem: *Suppose u is a continuous bounded function on $\bar{H}$ that is harmonic on H . Then u is the Poisson integral of its boundary values. More precisely,*

$$u = P_H[u|_{\mathbf{R}^{n-1}}]$$

on H .

We now take up the more general Poisson integrals defined earlier. Certain statements and proofs closely parallel those in the last chapter; we will be brief about details in these cases.

7.6 Theorem: *The following growth estimates apply to Poisson integrals:*

- (a) *If $\mu \in M(\mathbf{R}^{n-1})$ and $u = P_H[\mu]$, then $\|u_y\|_1 \leq \|\mu\|$ for all $y > 0$.*
- (b) *Suppose $1 \leq p \leq \infty$. If $f \in L^p(\mathbf{R}^{n-1})$ and $u = P_H[f]$, then $\|u_y\|_p \leq \|f\|_p$ for all $y > 0$.*

PROOF: The identity

$$7.7 \quad P_H((x, y), t) = P_H((t, y), x),$$

valid for all $x, t \in \mathbf{R}^{n-1}$ and $y > 0$, is the replacement for 6.5 in this context. The rest of the proof is the same as that of Theorem 6.4. $\square$

The next result is the upper half-space analogue of Theorem 6.6. Here the noncompactness of $\partial H = \mathbf{R}^{n-1}$ forces us to do a little extra work.

7.8 Theorem: Suppose that $1 \leq p < \infty$. If $f \in L^p(\mathbf{R}^{n-1})$ and $u = P_H[f]$, then $\|u_y - f\|_p \rightarrow 0$ as $y \rightarrow 0$.

PROOF: We first prove the theorem for $f \in C_c(\mathbf{R}^{n-1})$, the set of continuous functions on $\mathbf{R}^{n-1}$ with compact support. Because $C_c(\mathbf{R}^{n-1})$ is dense in $L^p(\mathbf{R}^{n-1})$ for $1 \leq p < \infty$, the approximation argument used in proving Theorem 6.6 (together with Theorem 7.6) will finish the proof.

Let $f \in C_c(\mathbf{R}^{n-1})$, and set $u = P_H[f]$. Choose a ball $B(0, R)$ that contains the support of f . Because f is uniformly continuous on $\mathbf{R}^{n-1}$, Theorem 7.4 implies that $u_y \rightarrow f$ uniformly on $\mathbf{R}^{n-1}$ as $y \rightarrow 0$. Thus to show $\|u_y - f\|_p \rightarrow 0$, we need only show

$$7.9 \quad \int_{|x| > 2R} |u_y(x)|^p dx \rightarrow 0$$

as $y \rightarrow 0$.

For $|x| > 2R$, we have

$$\begin{aligned} |u_y(x)|^p &\leq \int_{|t| < R} \frac{c_n y}{(|x - t|^2 + y^2)^{n/2}} |f(t)|^p dt \\ &\leq \frac{C y}{(|x| - R)^n}, \end{aligned}$$

where $C = c_n \|f\|_\infty^p V_{n-1}(B(0, R))$; the first inequality follows from Jensen's inequality. It is now easy to verify 7.9 by integrating in polar coordinates. $\square$

As in the last chapter, weak* convergence replaces norm convergence for Poisson integrals of measures and L^∞ -functions.

7.10 Theorem: Poisson integrals have the following weak* convergence properties:

- (a) If $\mu \in M(\mathbf{R}^{n-1})$ and $u = P_H[\mu]$, then $u_y \rightarrow \mu$ weak* in $M(\mathbf{R}^{n-1})$ as $y \rightarrow 0$.
- (b) If $f \in L^\infty(\mathbf{R}^{n-1})$ and $u = P_H[f]$, then $u_y \rightarrow f$ weak* in $L^\infty(\mathbf{R}^{n-1})$ as $y \rightarrow 0$.

PROOF: The Banach spaces $M(\mathbf{R}^{n-1})$ and $L^\infty(\mathbf{R}^{n-1})$ are, respectively, the dual spaces of $C_0(\mathbf{R}^{n-1})$ and $L^1(\mathbf{R}^{n-1})$. Note that if $g \in C_0(\mathbf{R}^{n-1})$, then g is uniformly continuous on $\mathbf{R}^{n-1}$, and therefore $(P_H[g])_y \rightarrow g$ uniformly on $\mathbf{R}^{n-1}$ as $y \rightarrow 0$ by Theorem 7.4. The proof of Theorem 6.8 can thus be used here, essentially verbatim. Again, the identity 7.7 replaces 6.5. $\square$

The Harmonic Hardy Spaces $h^p(H)$

For $p \in [1, \infty]$, we define the harmonic Hardy space $h^p(H)$ to be the normed vector space of functions u harmonic on H for which

$$\|u\|_{h^p} = \sup_{y>0} \|u_y\|_p < \infty.$$

Note that $h^\infty(H)$ is simply the collection of bounded harmonic functions on H , and that

$$\|u\|_{h^\infty} = \sup_{z \in H} |u(z)|.$$

We leave it to the reader to verify that $h^p(H)$ is a normed linear space under the norm $\|\cdot\|_{h^p}$.

Our work so far in this chapter easily yields the following results:

- (a) The map $\mu \mapsto P_H[\mu]$ is a linear isometry of $M(\mathbf{R}^{n-1})$ into $h^1(H)$.
- (b) For $1 < p \leq \infty$, the map $f \mapsto P_H[f]$ is a linear isometry of $L^p(\mathbf{R}^{n-1})$ into $h^p(H)$.

The proofs of (a) and (b) are the same as those of analogous assertions in the last chapter.

As in the last chapter, we wish to show that the maps in (a) and (b) are onto. The noncompactness of $\partial H = \mathbf{R}^{n-1}$ again requires us to do some extra work.

7.11 Theorem: *Let $p \in [1, \infty)$. Then there exists a constant C , depending only on p and n , such that*

$$|u(x, y)| \leq \frac{C \|u\|_{h^p}}{y^{(n-1)/p}}$$

for all $u \in h^p(H)$ and all $(x, y) \in H$. In particular, every $u \in h^p(H)$ is bounded on $H + (0, y)$ for each $y > 0$.

PROOF: Let $(x_0, y_0) \in H$, and let ω denote the open ball in $\mathbf{R}^n$ with center (x_0, y_0) and radius $y_0/2$. The volume version of the mean-value property, together with Jensen's inequality, shows that

$$\begin{aligned} 7.12 \quad |u(x_0, y_0)|^p &\leq \frac{1}{V_n(\omega)} \int_{\omega} |u|^p dV_n \\ &= \frac{2^n}{V_n(B)y_0^n} \int_{\omega} |u|^p dV_n. \end{aligned}$$

Setting $\Omega = \{(x, y) \in H : y_0/2 < y < 3y_0/2\}$, we have

$$\begin{aligned} \int_{\omega} |u|^p dV_n &\leq \int_{\Omega} |u|^p dV_n \\ &= \int_{y_0/2}^{3y_0/2} \int_{\mathbf{R}^{n-1}} |u(x, y)|^p dx dy \\ &\leq y_0 (\|u\|_{h^p})^p. \end{aligned}$$

This estimate and 7.12 give the conclusion of the theorem after taking p^{th} roots. $\square$

Theorems 7.5 and 7.11 give us the following important result: If $p \in [1, \infty]$ and $u \in h^p(H)$, then for each $y > 0$,

$$7.13 \quad u(z + (0, y)) = P_H[u_y](z)$$

for all $z \in H$. Equation 7.13 allows us to complete our description of the spaces $h^p(H)$.

7.14 Theorem: *The Poisson integral induces the following surjective isometries:*

- (a) *The map $\mu \mapsto P_H[\mu]$ is a linear isometry of $M(\mathbf{R}^{n-1})$ onto $h^1(H)$.*
- (b) *For $1 < p \leq \infty$, the map $f \mapsto P_H[f]$ is a linear isometry of $L^p(\mathbf{R}^{n-1})$ onto $h^p(H)$.*

PROOF: Because of 7.13, the proof of Theorem 6.12 carries over directly. $\square$

From the Ball to the Upper Half-Space, and Back

Recall the inversion map $x \mapsto x^*$ defined in Chapter 4. This map takes spheres containing 0 onto hyperplanes, and takes the interiors of such spheres onto open half-spaces. Composing the inversion map with appropriate translations and dilations will give us a one-to-one map of B onto H . There are many such maps; the one we choose below has the advantage of being its own inverse under composition.

Let $\mathbf{N} = (0, 1)$ and $\mathbf{S} = (0, -1)$ (here 0 denotes the origin in $\mathbf{R}^{n-1}$); we can think of $\mathbf{N}$ and $\mathbf{S}$ as the north and south poles of the unit sphere S . Now define $\Phi: \mathbf{R}^n \setminus \{\mathbf{S}\} \rightarrow \mathbf{R}^n \setminus \{\mathbf{S}\}$ by

$$\Phi(z) = 2(z - \mathbf{S})^* + \mathbf{S}.$$

It is easy to see that Φ is a one-to-one map of $\mathbf{R}^n \setminus \{\mathbf{S}\}$ onto itself. We can regard Φ as a homeomorphism of $\mathbf{R}^n \cup \{\infty\}$ onto itself by defining $\Phi(\mathbf{S}) = \infty$ and $\Phi(\infty) = \mathbf{S}$. The next result summarizes the basic properties of Φ that we need.

7.15 Proposition: *The map Φ has the following properties:*

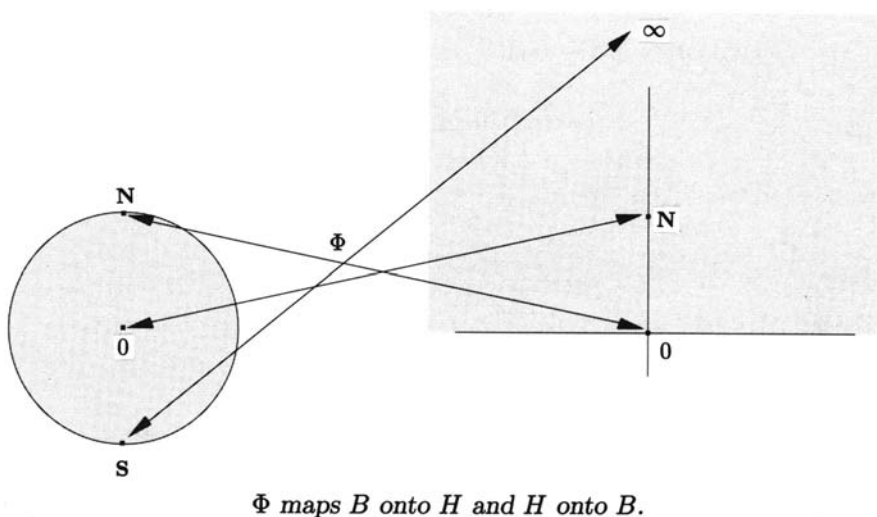
- (a) $\Phi(\Phi(z)) = z$ for all $z \in \mathbf{R}^n \cup \{\infty\}$;
- (b) Φ is a conformal, one-to-one map of $\mathbf{R}^n \setminus \{\mathbf{S}\}$ onto $\mathbf{R}^n \setminus \{\mathbf{S}\}$;
- (c) Φ maps B onto H and H onto B ;
- (d) Φ maps $S \setminus \{\mathbf{S}\}$ onto $\mathbf{R}^{n-1}$ and $\mathbf{R}^{n-1}$ onto $S \setminus \{\mathbf{S}\}$.

PROOF: The proof of (a) is a simple computation.

In (b), only conformality needs to be checked. Recalling that the inversion map is conformal (Lemma 4.2), we see that Φ is the composition of conformal maps, and hence is itself conformal.

We prove (c) and (d) together. Noting that $\Phi(\mathbf{S}) = \infty$, we know that Φ maps $S \setminus \{\mathbf{S}\}$ onto some hyperplane. Because the inversion map preserves the $(0, y)$ -axis, the same is true Φ . The conformality of Φ thus shows that $\Phi(S \setminus \{\mathbf{S}\})$ is a hyperplane perpendicular to the $(0, y)$ -axis. Since $\Phi(\mathbf{N}) = 0$, we must have $\Phi(S \setminus \{\mathbf{S}\}) = \mathbf{R}^{n-1}$. It follows that $\Phi(B)$ is either the upper or lower half-space. Because $\Phi(0) = \mathbf{N}$, we have $\Phi(B) = H$, as desired. $\square$

The reader may find it helpful to keep the following diagram in mind as we proceed.



We now introduce a modification of the Kelvin transform that will take harmonic functions on B to harmonic functions on H and vice-versa. Given any function u defined on a set $E \subset \mathbf{R}^n \setminus \{\mathbf{S}\}$, we define the function $\mathcal{K}[u]$ on $\Phi(E)$ by

$$\mathcal{K}[u](z) = 2^{(n-2)/2} |z - \mathbf{S}|^{2-n} u(\Phi(z)).$$

Note that when $n = 2$, $\mathcal{K}[u](z) = u(\Phi(z))$.

The factor $2^{(n-2)/2}$ is included so that $\mathcal{K}$ will be its own inverse. That is, we claim

$$\mathcal{K}[\mathcal{K}[u]] = u$$

for all u as above, a computation we leave to the reader.

The transform $\mathcal{K}$ is linear: If u, v are functions on E and b, c are constants, then

$$\mathcal{K}[bu + cv] = b\mathcal{K}[u] + c\mathcal{K}[v]$$

on $\Phi(E)$.

Finally, $\mathcal{K}$ preserves harmonicity. The real work for the proof of this was done when we proved Theorem 4.4.

7.16 Proposition: *If $\Omega \subset \mathbf{R}^n \setminus \{\mathbf{S}\}$, then u is harmonic on Ω if and only if $\mathcal{K}[u]$ is harmonic on $\Phi(\Omega)$.*

PROOF: Because $\mathcal{K}$ is its own inverse, it suffices to prove only one direction of the theorem. So suppose that u is harmonic on Ω . Define a harmonic function v on $\frac{1}{2}(\Omega - \mathbf{S})$ by $v(z) = u(2z + \mathbf{S})$. By Theorem 4.4, the Kelvin transform $K[v]$ is harmonic on $2(\Omega - \mathbf{S})^*$, and thus $K[v](z - \mathbf{S})$ is harmonic on $2(\Omega - \mathbf{S})^* + \mathbf{S} = \Phi(\Omega)$. But, as is easily checked, $K[v](z - \mathbf{S}) = 2^{(2-n)/2}\mathcal{K}[u](z)$, so that $\mathcal{K}[u]$ is harmonic on $\Phi(\Omega)$, as desired. $\square$

Positive Harmonic Functions on the Upper Half-Space

Because the $\mathcal{K}$ -transform takes positive functions to positive functions, Proposition 7.16 shows that $\mathcal{K}$ preserves the class of positive harmonic functions. As a consequence, u is positive and harmonic on H if and only if $\mathcal{K}[u]$ is positive and harmonic on B . This important fact will allow us to transfer our knowledge about positive harmonic functions on the ball to the upper half-space. For example, we can now prove an analogue of Theorem 6.15 for the upper half-space.

7.17 Theorem: *Let $t \in \mathbf{R}^{n-1}$. Suppose that u is positive and harmonic on H , and that u extends continuously to $\bar{H} \setminus \{t\}$ with boundary values 0 on $\mathbf{R}^{n-1} \setminus \{t\}$. Suppose further that*

$$7.18 \quad \frac{u(0, y)}{y} \rightarrow 0 \quad \text{as } y \rightarrow \infty.$$

Then $u = cP_H(\cdot, t)$ for some positive constant c .

PROOF: The function $\mathcal{K}[u]$ is positive and harmonic in B . Thus by 6.15,

$$\mathcal{K}[u] = P_B[\mu]$$

for some positive $\mu \in M(S)$, where P_B denotes the Poisson kernel for the ball. Our hypothesis on u implies that $\mathcal{K}[u]$ extends continuously to $\bar{B} \setminus \{S, \Phi(t)\}$, with boundary values 0 on $S \setminus \{S, \Phi(t)\}$. The argument used in proving Theorem 6.15 shows then that μ is the sum of point masses at S and $\Phi(t)$.

An easy computation gives

$$\mathcal{K}[u](rS) = 2^{(n-2)/2}(1-r)^{2-n}u\left(0, \frac{1+r}{1-r}\right)$$

for any $r \in [0, 1)$. From 7.18 we then see that $(1-r)^{n-1}\mathcal{K}[u](rS) \rightarrow 0$ as $r \rightarrow 1$, and this implies $\mu(\{S\}) = 0$ (see Exercise 3 in Chapter 6).

Thus μ is a point mass at $\Phi(t)$, and therefore $\mathcal{K}[u]$ is a constant times $P_B(\cdot, \Phi(t))$. Because $P_H(\cdot, t)$ also satisfies the hypotheses of Theorem 7.17, $\mathcal{K}[P_H(\cdot, t)]$ is a constant times $P_B(\cdot, \Phi(t))$ as well. Thus

$$\mathcal{K}[u] = c\mathcal{K}[P_H(\cdot, t)]$$

for some positive constant c . Applying $\mathcal{K}$ to both sides of the last equation, we see that the linearity of $\mathcal{K}$ gives the conclusion of the theorem. $\square$

We can think of the next result as the “ $t = \infty$ ” case of Theorem 7.17.

7.19 Theorem: *Suppose that u is positive and harmonic on H , and that u extends continuously to $\bar{H}$ with boundary values 0 on $\mathbf{R}^{n-1}$. Then there exists a positive constant c such that $u(x, y) = cy$ for all $(x, y) \in H$.*

PROOF: The function $\mathcal{K}[u]$ is positive and harmonic in B , extends continuously to $\bar{B} \setminus \{\mathbf{S}\}$, and has boundary values 0 on $S \setminus \{\mathbf{S}\}$. By Theorem 6.15, $\mathcal{K}[u]$ is a constant times $P_B(\cdot, \mathbf{S})$. Because the same is true of $\mathcal{K}[v]$, where $v(x, y) = y$ on H , $\mathcal{K}[u]$ is a constant times $\mathcal{K}[v]$. As in the proof of the last theorem, this gives us the desired conclusion. $\square$

The $\mathcal{K}$ -transform allows us to derive the relation between P_B and P_H , the Poisson kernels for B and H , with a minimum of computation.

7.20 Theorem: For all $z \in H$ and $t \in \mathbf{R}^{n-1}$,

$$7.21 \quad P_H(z, t) = 2^{n-2} c_n (1 + |t|^2)^{-n/2} |z - \mathbf{S}|^{2-n} P_B(\Phi(z), \Phi(t)).$$

PROOF: Fix $t \in \mathbf{R}^{n-1}$, and let $u(z)$ denote the right side of 7.21. Then u is positive and harmonic on H , and it is easy to check that u extends continuously to $\bar{H} \setminus \{t\}$ with boundary values 0 on $\mathbf{R}^{n-1} \setminus \{t\}$. We also see that $u(0, y)/y \rightarrow 0$ (with plenty of room to spare) as $y \rightarrow \infty$. Thus by Theorem 7.17, u is a constant multiple of $P_H(\cdot, t)$. Evaluating at $z = \mathbf{N}$ now gives 7.21. $\square$

We turn now to the problem of characterizing the positive harmonic functions on H . We know that if μ is a finite positive Borel measure on $\mathbf{R}^{n-1}$, then $P_H[\mu]$ is a positive harmonic function on H . Unlike the case for the ball, however, not all positive harmonic functions on H arise in this manner. In the first place, $P_H[\mu]$ defines a positive harmonic function on H for some positive measures μ that are not finite—Lebesgue measure on $\mathbf{R}^{n-1}$, for example. Secondly, the positive harmonic function y is not the Poisson integral of anything that lives on $\mathbf{R}^{n-1}$.

Let us note that if μ is any positive Borel measure on $\mathbf{R}^{n-1}$, then

$$7.22 \quad P_H[\mu](z) = \int_{\mathbf{R}^{n-1}} P_H(z, t) d\mu(t)$$

is well-defined as a number in $[0, \infty]$ for every $z \in H$. We claim that 7.22 defines a positive harmonic function on H precisely when

$$7.23 \quad \int_{\mathbf{R}^{n-1}} \frac{d\mu(t)}{(1 + |t|^2)^{n/2}} < \infty.$$

To see this, note that for any fixed $z \in H$, $P_H(z, t)$, as a function of t , is bounded above and below by positive constant multiples of $(1 + |t|^2)^{-n/2}$. Thus if 7.22 is finite for some $z \in H$, then it is finite for all $z \in H$, and this happens exactly when 7.23 occurs. In this case $P_H[\mu]$ is harmonic on H , as can be verified by checking the volume version of the mean-value property.

We now state the main result of this section.

7.24 Theorem: *If u is positive and harmonic on H , then there exists a positive Borel measure μ on $\mathbf{R}^{n-1}$ and a nonnegative constant c such that*

$$u(x, y) = P_H[\mu](x, y) + cy$$

for all $(x, y) \in H$.

The main idea in the proof of this result is simple: If u is positive and harmonic on H , then $\mathcal{K}[u]$ is positive and harmonic on B , and hence is the Poisson integral of a positive measure on S . The restriction of this measure to $S \setminus \{S\}$ gives rise to the measure μ , and the mass of this measure at S gives rise to the term cy .

Before coming to the proof of Theorem 7.24 proper, we need to understand how measures on S pull back, via the map Φ , to measures on $\mathbf{R}^{n-1}$. For any positive $\nu \in M(S)$, we can define a positive measure $\nu \circ \Phi \in M(\mathbf{R}^{n-1})$ by setting $(\nu \circ \Phi)(E) = \nu(\Phi(E))$ for every Borel set $E \subset \mathbf{R}^{n-1}$. We then have the following “change of variables formula”, valid for every positive Borel measurable function f on $S \setminus \{S\}$:

$$7.25 \quad \int_{S \setminus \{S\}} f d\nu = \int_{\mathbf{R}^{n-1}} (f \circ \Phi) d(\nu \circ \Phi).$$

The last equation is easy to verify when f is a simple function on $S \setminus \{S\}$; the full result follows from this by the monotone convergence theorem.

PROOF OF THEOREM 7.24: If u is positive and harmonic on H , then $\mathcal{K}[u]$ is positive and harmonic on B , and thus $\mathcal{K}[u] = P_B[\lambda]$ for some positive measure $\lambda \in M(S)$. Define $\nu \in M(S)$ by $d\nu = \chi_{S \setminus \{S\}} d\lambda$. We then have

$$\mathcal{K}[u] = P_B[\nu] + \lambda(\{S\})P_B(\cdot, S).$$

By the linearity of $\mathcal{K}$,

$$u = \mathcal{K}[P_B[\nu]] + \lambda(\{\mathbf{S}\})\mathcal{K}[P_B(\cdot, \mathbf{S})].$$

From Theorem 7.19 it is easy to see that $\mathcal{K}[P_B(\cdot, \mathbf{S})]$ is a constant multiple of y on H . The proof will be completed by showing that $\mathcal{K}[P_B[\nu]] = P_H[\mu]$ for some positive Borel measure μ on $\mathbf{R}^{n-1}$.

Because $\nu(\{\mathbf{S}\}) = 0$,

$$P_B[\nu](z) = \int_{S \setminus \{\mathbf{S}\}} P_B(z, \zeta) d\nu(\zeta)$$

for all $z \in B$. Thus by 7.25,

$$\begin{aligned} \mathcal{K}[P_B[\nu]](z) &= \int_{S \setminus \{\mathbf{S}\}} 2^{(n-2)/2} |z - \mathbf{S}|^{2-n} P_B(\Phi(z), \zeta) d\nu(\zeta) \\ &= \int_{\mathbf{R}^{n-1}} 2^{(n-2)/2} |z - \mathbf{S}|^{2-n} P_B(\Phi(z), \Phi(t)) d(\nu \circ \Phi)(t) \end{aligned}$$

for every $z \in H$. In the last integral we may multiply and divide by $\psi(t)$, where $\psi(t) = 2^{(n-2)/2} c_n(1+|t|^2)^{-n/2}$. With $d\mu = (1/\psi) d(\nu \circ \Phi)$, we then have $\mathcal{K}[P_B[\nu]] = P_H[\mu]$, as desired. $\square$

Nontangential Limits

We now look briefly at the Fatou Theorem for the Poisson integrals discussed in this chapter. Rather than tediously verifying that the maximal function arguments of the last chapter carry through to the present setting, we use the $\mathcal{K}$ -transform to transfer the Fatou Theorem from B to H .

The notion of a nontangential limit for a function on H was defined in Chapter 2; the analogous definition for a function on B was given in Chapter 6. We leave it to the reader to verify the following assertion, which follows from the conformality of the map Φ : A function u on H has a nontangential limit at $t \in \mathbf{R}^{n-1}$ if and only if $\mathcal{K}[u]$ has a nontangential limit (within B) at $\Phi(t)$.

Another observation that we leave to the reader is that the map Φ preserves sets of measure zero. More precisely, a Borel set $E \subset \mathbf{R}^{n-1}$ has Lebesgue measure 0 if and only if $\Phi(E)$ has σ -measure 0 on S ; this follows easily from the smoothness of Φ .

In this chapter, the term “almost everywhere” will refer to Lebesgue measure on $\mathbf{R}^{n-1}$. Putting the last two observations together, we see that a function u on H has nontangential limits almost everywhere on $\mathbf{R}^{n-1}$ if and only if $\mathcal{K}[u]$ has nontangential limits σ -almost everywhere on S .

The next result is the Fatou theorem for Poisson integrals of functions in $L^p(\mathbf{R}^{n-1})$.

7.26 Theorem: *Let $p \in [1, \infty]$. If $f \in L^p(\mathbf{R}^{n-1})$, then $P_H[f]$ has nontangential limit $f(x)$ at almost every $x \in \mathbf{R}^{n-1}$.*

PROOF: Because every real-valued function in $L^p(\mathbf{R}^{n-1})$ is the difference of two positive functions in $L^p(\mathbf{R}^{n-1})$, we may assume that $f \geq 0$. The function $u = P_H[f]$ is then positive and harmonic on H , and thus $\mathcal{K}[u]$ is positive and harmonic on B . By 6.14 and 6.36, $\mathcal{K}[u]$ has nontangential limits σ -almost everywhere on S . As observed earlier, this implies that u has a nontangential limit $g(x)$ for almost every $x \in \mathbf{R}^{n-1}$.

We need to verify that $f = g$ almost everywhere. For $p < \infty$, Theorem 7.8 asserts that $\|u_y - f\|_p \rightarrow 0$ as $y \rightarrow 0$; thus some subsequence (u_{y_k}) converges to f pointwise almost everywhere on $\mathbf{R}^{n-1}$, and hence $f = g$. For $p = \infty$, Theorem 7.10(b) shows that $u_y \rightarrow f$ weak* in $L^\infty(\mathbf{R}^{n-1})$ as $y \rightarrow 0$. But we also have $u_y \rightarrow g$ weak* in $L^\infty(\mathbf{R}^{n-1})$ by the dominated convergence theorem, and so we conclude $f = g$. $\square$

The theorem above shows, by Theorem 7.14, that if $u \in h^p(H)$ and $p \in (1, \infty]$, then u has nontangential limits almost everywhere on $\mathbf{R}^{n-1}$. The next theorem gives us the same result for $h^1(H)$ as a corollary.

7.27 Theorem: *If $\mu \in M(\mathbf{R}^{n-1})$ is singular with respect to Lebesgue measure, then $P_H[\mu]$ has nontangential limit 0 almost everywhere on $\mathbf{R}^{n-1}$.*

PROOF: We may assume that μ is positive. By analogy with 7.25, we define $\mu \circ \Phi \in M(S)$ by setting $(\mu \circ \Phi)(E) = \mu(\Phi(E \setminus \{S\}))$ for every Borel set $E \subset S$. We then have

$$\mathcal{K}[P_B[\mu \circ \Phi]] = P_H[\nu],$$

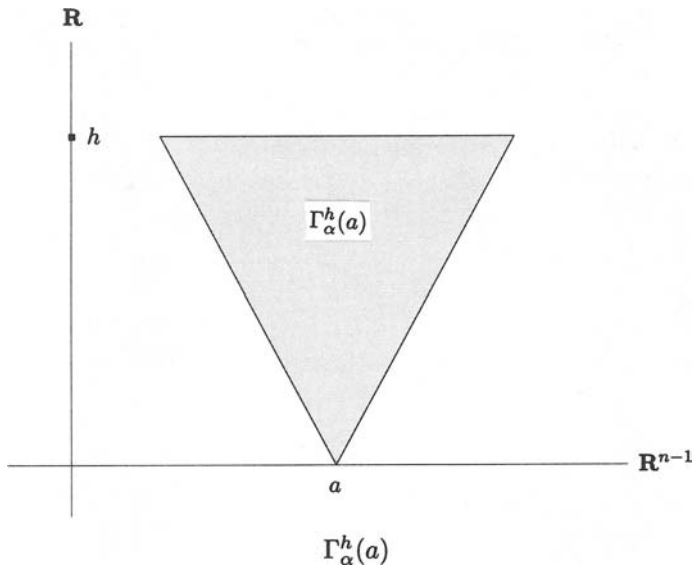
where $d\nu = (1/\psi) d\mu$ and ψ is as in the proof of 7.24. Because μ is singular with respect to Lebesgue measure on $\mathbf{R}^{n-1}$, $\mu \circ \Phi$ is singular with respect to σ . By 6.34, $P_B[\mu \circ \Phi]$ has nontangential limit 0 almost everywhere on S . The equation above tells us that $P_H[\nu]$ has nontangential limit 0 almost everywhere on $\mathbf{R}^{n-1}$. From this we easily deduce that $P_H[\mu]$ has nontangential limit 0 almost everywhere on $\mathbf{R}^{n-1}$. $\square$

The Local Fatou Theorem

The Fatou Theorems obtained so far in this book apply to Poisson integrals of functions or measures. In this section we prove a different kind of Fatou theorem—one that applies to arbitrary harmonic functions on H satisfying a certain local boundedness condition.

We will need to consider truncations of the cones $\Gamma_\alpha(a)$ defined in Chapter 2. Thus, for any $h > 0$, we define

$$\Gamma_\alpha^h(a) = \{(x, y) \in H : |x - a| < \alpha y \text{ and } y < h\}.$$



A function u on H is said to be *nontangentially bounded* at

$a \in \mathbf{R}^{n-1}$ if u is bounded in some $\Gamma_\alpha^h(a)$. Note that if u is continuous on H , then u is nontangentially bounded at a if and only if u is bounded in $\Gamma_\alpha^1(a)$ for some $\alpha > 0$.

We can now state the main result of this section.

7.28 The Local Fatou Theorem: *Suppose u is harmonic on H and $E \subset \mathbf{R}^{n-1}$ is the set of points at which u is nontangentially bounded. Then u has a nontangential limit at almost every point of E .*

A remarkable feature of this theorem should be stressed: For each $a \in E$, we are only assuming that u is bounded in some $\Gamma_\alpha^h(a)$; in particular, α can depend on a . Nevertheless, the theorem asserts the existence of a set of full measure $F \subset E$ such that u has a limit in $\Gamma_\alpha(a)$ for every $a \in F$ and every $\alpha > 0$.

The following lemma will be important in proving the Local Fatou Theorem. Figure 7.30 may be helpful in picturing the geometry of the region Ω mentioned in the next three lemmas.

7.29 Lemma: *Suppose $E \subset \mathbf{R}^{n-1}$ is Borel measurable, $\alpha > 0$, and*

$$\Omega = \bigcup_{a \in E} \Gamma_\alpha^1(a).$$

Then there exists a positive harmonic function v on H such that $v \geq 1$ on $(\partial\Omega) \cap H$, and such that v has nontangential limit 0 almost everywhere on E .

PROOF: Define a positive harmonic function w on H by

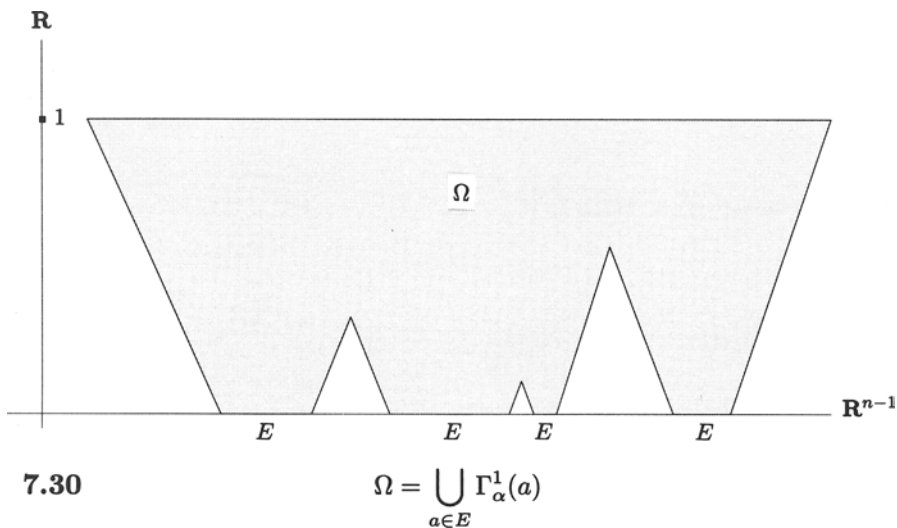
$$w(x, y) = P_H[\chi_{E^c}](x, y) + y,$$

where χ_{E^c} denotes the characteristic function of E^c , the complement of E in $\mathbf{R}^{n-1}$. By Theorem 7.26, w has nontangential limit 0 almost everywhere on E .

We wish to show that w is bounded away from 0 on $(\partial\Omega) \cap H$. Because $w(x, 1) \geq 1$, we have $w \geq 1$ on the “top” of $\partial\Omega$. Next, observe that (x, y) belongs to $\Gamma_\alpha(a)$ if and only if $a \in B(x, \alpha y)$ (where $B(x, \alpha y)$ denotes the ball in $\mathbf{R}^{n-1}$ with center x and radius αy). So if $(x, y) \in \partial\Omega$ and $0 < y < 1$, then $(x, y) \notin \Gamma_\alpha(a)$ for all $a \in E$ (otherwise $(x, y) \in \Omega$), giving $B(x, \alpha y) \subset E^c$. Therefore

$$\begin{aligned}
P_H[\chi_{E^c}](x, y) &= \int_{E^c} \frac{c_n y}{(|x - t|^2 + y^2)^{n/2}} dt \\
&\geq \int_{B(x, \alpha y)} \frac{c_n y}{(|x - t|^2 + y^2)^{n/2}} dt \\
&= \int_{B(0, \alpha)} \frac{c_n}{(|t|^2 + 1)^{n/2}} dt.
\end{aligned}$$

Denoting the last expression by c_α (a constant less than 1 that depends only on α and n), we see that if $v = w/c_\alpha$, then v satisfies the conclusion of the lemma. $\square$



The crux of the proof of 7.28 is the following weaker version of the Local Fatou Theorem.

7.31 Lemma: Let $E \subset \mathbf{R}^{n-1}$ be Borel measurable, let $\alpha > 0$, and let

$$\Omega = \bigcup_{a \in E} \Gamma_\alpha^1(a).$$

Suppose u is harmonic on H and bounded on Ω . Then for almost every $a \in E$, $\lim u(z)$ exists as $z \rightarrow a$ within $\Gamma_\alpha(a)$.

PROOF: Because every Borel set can be written as a countable union of bounded Borel sets, we may assume E is bounded. We may also assume that u is real valued.

Because u is continuous on H and E is bounded, we may assume that $|u| \leq 1$ on the open set

$$\Omega' = \bigcup_{a \in E} \Gamma_a^2(a).$$

Choose a sequence (y_k) in the interval $(0, 1)$ such that $y_k \rightarrow 0$, and set $E_k = [\Omega - (0, y_k)] \cap \mathbf{R}^{n-1}$. Each E_k is an open subset of $\mathbf{R}^{n-1}$ that contains E . (At this point we suggest the reader start drawing some pictures.)

For $x \in \mathbf{R}^{n-1}$, define

$$f_k(x) = \chi_{E_k}(x)u(x, y_k).$$

Because $(x, y_k) \in \Omega$ if and only if $x \in E_k$, $|f_k| \leq 1$ on $\mathbf{R}^{n-1}$ for every k . Being norm-bounded in $L^\infty(\mathbf{R}^{n-1})$, there exists a subsequence, which we still denote by (f_k) , that converges weak* to some $f \in L^\infty(\mathbf{R}^{n-1})$.

Now each f_k is continuous on E_k (because E_k is open), and thus $P_H[f_k]$ extends continuously to $H \cup E_k$ (see Exercise 16(a) of this chapter). The function u_k given by

$$u_k(x, y) = P_H[f_k](x, y) - u(x, y + y_k)$$

is thus harmonic on H and extends continuously to $H \cup E_k$, with $u_k = 0$ on E_k . In particular, u_k is continuous on $\bar{\Omega}$ with $u_k = 0$ on E . Furthermore, because $\Omega + (0, y_k) \subset \Omega'$, we have $|u_k| \leq 2$ on $\bar{\Omega}$.

Now let v denote the function of Lemma 7.29 with respect to Ω . Then $\liminf_{z \rightarrow \partial\Omega} (2v - u_k)(z) \geq 0$. By the minimum principle, $2v - u_k \geq 0$ in Ω . Letting $k \rightarrow \infty$, we then see that $2v - (P_H[f] - u) \geq 0$ in Ω . Because this argument applies as well to $2v + u_k$, we conclude that $|P_H[f] - u| \leq 2v$ in Ω .

By Theorem 7.26, $P_H[f]$ has nontangential limits almost everywhere on $\mathbf{R}^{n-1}$, while Lemma 7.29 asserts v has nontangential limits 0 almost everywhere on E . From this and the last inequality, the desired limits for u follow. $\square$

Recall that if $E \subset \mathbf{R}^{n-1}$ is Borel measurable, then a point $a \in E$ is said to be a *point of density* of E provided

$$\lim_{r \rightarrow 0} \frac{V_{n-1}(B(a, r) \cap E)}{V_{n-1}(B(a, r))} = 1.$$

By the Lebesgue Differentiation Theorem ([9], Theorem 7.7), almost every point of E is a point of density of E .

Points of density of E are where we can expect the cones defining Ω in Proposition 7.31 to “pile up”; this will allow us to pass from 7.31 to the stronger assertion in 7.28.

7.32 Lemma: Suppose $E \subset \mathbf{R}^{n-1}$ is Borel measurable, $\alpha > 0$, and

$$\Omega = \bigcup_{a \in E} \Gamma_{\alpha}^1(a).$$

Suppose u is continuous on H and bounded on Ω . If a is a point of density of E , then u is bounded in $\Gamma_{\beta}^1(a)$ for every $\beta > 0$.

PROOF: Let a be a point of density of E , and let $\beta > 0$. It suffices to show that $\Gamma_{\beta}^h(a) \subset \Omega$ for some $h > 0$.

Choose $\delta > 0$ such that

$$\mathbf{7.33} \quad \frac{V_{n-1}(B(a, r) \cap E)}{V_{n-1}(B(a, r))} > 1 - \left(\frac{\alpha}{\alpha + \beta} \right)^{n-1}$$

whenever $r < \delta$; we may assume $\delta/(\alpha + \beta) < 1$. Set $h = \delta/(\alpha + \beta)$, and let $(x, y) \in \Gamma_{\beta}^h(a)$. Then $B(x, \alpha y) \subset B(a, (\alpha + \beta)y)$. This implies $B(x, \alpha y) \cap E$ is non-empty; otherwise we violate 7.33 (take $r = (\alpha + \beta)y$). Choosing any $b \in B(x, \alpha y) \cap E$, we have $(x, y) \in \Gamma_{\alpha}^1(b)$, and thus $\Gamma_{\beta}^h(a) \subset \Omega$, as desired. $\square$

PROOF OF THEOREM 7.28: We are assuming u is harmonic on H and E is the set of points in $\mathbf{R}^{n-1}$ at which u is nontangentially bounded. For $k = 1, 2, \dots$, set $E_k = \{a \in \mathbf{R}^{n-1} : |u| \leq k \text{ on } \Gamma_{1/k}^1(a)\}$. Then each E_k is a closed subset of $\mathbf{R}^{n-1}$, and $E = \bigcup E_k$ (incidentally proving that the set E is Borel measurable). Applying Lemma 7.32 to each E_k , and recalling that the points of density of E_k form a set of full measure in E_k , we see that there is a set of full measure $F \subset E$ such that u is bounded in $\Gamma_{\alpha}^1(a)$ for every $a \in F$ and every $\alpha > 0$. For each positive integer k , we can write F as $F = \bigcup F_j$, where $F_j = \{a \in F : |u| \leq j \text{ on } \Gamma_k^1(a)\}$. Lemma 7.31, applied to F_j , now shows that u has nontangential limits almost everywhere on E , as desired. $\square$

Exercises

1. Assume $n = 2$. For each $t \in \mathbf{R}$, find a holomorphic function g_t on H such that $P_H(\cdot, t) = \operatorname{Re} g_t$.
2. In Chapter 1, we calculated $P_B(x, \zeta)$ as a normal derivative on ∂B of an appropriate modification of $|x - \zeta|^{2-n}$ ($n > 2$). Using an appropriate modification of $|z - t|^{2-n}$, find a function whose normal derivative on ∂H is $P_H(z, t)$.
3. Let $\mu \in M(\mathbf{R}^{n-1})$ and let $u = P_H[\mu]$. Prove that

$$\int_{B_{n-1}} u_y(x) dx \rightarrow \mu(B_{n-1}) + \frac{\mu(\partial B_{n-1})}{2}$$

as $y \rightarrow 0$.

4. Let $p \in [1, \infty]$ and assume $u \in h^p(H)$. Show that if the functions u_y converge uniformly on $\mathbf{R}^{n-1}$ as $y \rightarrow 0$, then u extends to a bounded uniformly continuous function on $\bar{H}$.
5. For $\zeta \in S$, show that

$$\Phi(\zeta) = \frac{(\zeta_1, \zeta_2, \dots, \zeta_{n-1}, 0)}{1 + \zeta_n}.$$

6. For $(x, y) \in \mathbf{R}^n \setminus \{S\}$, show that

$$1 - |\Phi(x, y)|^2 = \frac{4y}{|x|^2 + (y+1)^2}.$$

7. Show that if $n = 2$, then

$$\Phi(z) = \frac{1 - i\bar{z}}{\bar{z} - i}$$

for every $z \in \mathbf{C} \setminus \{-i\}$.

8. Prove that

$$\int_S f \, d\sigma = c_n 2^{n-2} \int_{\mathbf{R}^{n-1}} f(\Phi(t))(1 + |t|^2)^{1-n} \, dt.$$

for every positive Borel measurable function f on S . (*Hint:* Because $\Phi: S \setminus \{S\} \rightarrow \mathbf{R}^{n-1}$ is smooth, there exists a smooth function w on $\mathbf{R}^{n-1}$ such that $d(\sigma \circ \Phi) = w \, dt$. To find w , apply 7.25 with $\nu = \sigma$, and consider what happens when z tends to a point of $\mathbf{R}^{n-1}$.)

9. Using the result of the last exercise, show that

$$\int_{\mathbf{R}^{n-1}} f(t) \, dt = \frac{2}{c_n} \int_S f(\Phi(\zeta))(\zeta_n + 1)^{1-n} \, d\sigma(\zeta)$$

for every positive Borel measurable function f on $\mathbf{R}^{n-1}$.

10. (a) Let μ be a positive Borel measure on $\mathbf{R}^{n-1}$ that satisfies 7.23, and set $u = P_H[\mu]$. Show that $\lim_{y \rightarrow \infty} u(0, y)/y = 0$.
 (b) Let u be a positive harmonic function on H . Show that $\liminf_{y \rightarrow 0} u(0, y)/y > 0$.
11. Show that if u is positive and harmonic on H , then the decomposition $u(x, y) = P_H[\mu](x, y) + cy$ of Theorem 7.24 holds for a unique positive Borel measure μ on $\mathbf{R}^{n-1}$ and a unique non-negative constant c .
12. Let μ be a positive Borel measure on $\mathbf{R}^{n-1}$ that satisfies 7.23, and set $u = P_H[\mu]$. Prove that

$$\lim_{y \rightarrow 0} \int_{\mathbf{R}^{n-1}} \varphi(t) u_y(t) \, dt = \int_{\mathbf{R}^{n-1}} \varphi(t) \, d\mu(t)$$

for every continuous function φ on $\mathbf{R}^{n-1}$ with compact support.

13. Prove that $\mathcal{K}[h^p(H)] \subset h^1(B)$ for every $p \in [1, \infty]$. (*Hint:* Exercise 8 in Chapter 6 may be helpful here.)
14. Let $p \in [1, \infty]$ and let $f \in L^p(\mathbf{R}^{n-1})$. Show that $\mathcal{K}[f] \in L^1(S)$.

15. Let $p \in [1, \infty]$ and let $f \in L^p(\mathbf{R}^{n-1})$. Show that

$$\mathcal{K}\left[P_B[\mathcal{K}[f]]\right](z) = P_H[f](z)$$

for every $z \in H$.

16. Assume that f is measurable on $\mathbf{R}^{n-1}$ and that

$$\int_{\mathbf{R}^{n-1}} |f(t)|(1 + |t|^2)^{-n/2} dt < \infty.$$

- (a) Show that if f is continuous at a , then $P_H[f] \rightarrow f(a)$ as $z \rightarrow a$ within H .
- (b) Show that $P_H[f]$ tends nontangentially to f almost everywhere on $\mathbf{R}^{n-1}$. (*Hint:* Let g denote f times the characteristic function of some large ball. Apply Theorem 7.26 to $P_H[g]$; apply part (a) to $P_H[f - g]$.)
17. Let μ be a positive Borel measure on $\mathbf{R}^{n-1}$ that satisfies 7.23. Show that if μ is singular with respect to Lebesgue measure, then $P_H[\mu]$ has nontangential limit 0 at almost every point of $\mathbf{R}^{n-1}$.

CHAPTER 8

Harmonic Bergman Spaces

Throughout this chapter, p denotes a number satisfying $1 \leq p < \infty$. The Bergman space $b^p(\Omega)$ is the set of harmonic functions u on Ω such that

$$\|u\|_p = \left(\int_{\Omega} |u|^p dV \right)^{1/p} < \infty.$$

We often view $b^p(\Omega)$ as a subspace of $L^p(\Omega, dV)$. The spaces $b^p(\Omega)$ are named in honor of Stefan Bergman, who studied analogous spaces of holomorphic functions.

For fixed $x \in \Omega$, the map $u \mapsto u(x)$ is a linear functional on $b^p(\Omega)$; we refer to this map as *point evaluation* at x . The following proposition shows that point evaluation is continuous on $b^p(\Omega)$.

8.1 Proposition: *Suppose $x \in \Omega$. Then*

$$|u(x)| \leq \frac{1}{V(B)^{1/p} d(x, \partial\Omega)^{n/p}} \|u\|_p$$

for every $u \in b^p(\Omega)$.

PROOF: Let r be a positive number with $r < d(x, \partial\Omega)$, and apply the volume version of the mean-value property to u on $B(x, r)$. After taking absolute values, Jensen's inequality gives

$$|u(x)|^p \leq \frac{1}{V(B(x, r))} \int_{B(x, r)} |u|^p dV \leq \frac{1}{r^n V(B)} \|u\|_p^p.$$

The desired inequality is now obtained by taking p^{th} roots and letting $r \rightarrow d(x, \partial\Omega)$. $\square$

8.2 Corollary: *For every multi-index α there exists a constant C_α such that*

$$|D^\alpha u(x)| \leq \frac{C_\alpha}{d(x, \partial\Omega)^{|\alpha|+n/p}} \|u\|_p$$

for all $x \in \Omega$ and every $u \in b^p(\Omega)$.

PROOF: Apply 8.1 and Cauchy's estimates (2.4) to u on the ball of radius $d(x, \partial\Omega)/2$ centered at x . $\square$

The next result shows that $b^p(\Omega)$ is a Banach space.

8.3 Proposition: *$b^p(\Omega)$ is a closed subspace of $L^p(\Omega, dV)$.*

PROOF: Suppose $\|u_j - u\|_p \rightarrow 0$, where (u_j) is a sequence in $b^p(\Omega)$ and $u \in L^p(\Omega, dV)$. We must show that, after appropriate modification on a set of measure zero, u is harmonic on Ω .

Let $K \subset \Omega$ be compact. By Proposition 8.1, there is a constant $C < \infty$ such that $|u_j(x) - u_k(x)| \leq C\|u_j - u_k\|_p$ for all $x \in K$ and all j, k . Because (u_j) is Cauchy in $b^p(\Omega)$, this implies that (u_j) converges uniformly on K . Thus (u_j) converges uniformly on compact subsets of Ω to a function v that is harmonic on Ω (Theorem 1.19).

Because $u_j \rightarrow u$ in $L^p(\Omega, dV)$, some subsequence of (u_j) converges to u pointwise almost everywhere on Ω . It follows that $u = v$ almost everywhere on Ω , and thus $u \in b^p(\Omega)$, as desired. $\square$

Reproducing Kernels

Taking $p = 2$, we see that the last proposition shows that $b^2(\Omega)$ is a Hilbert space with inner product

$$\langle u, v \rangle = \int_{\Omega} u \bar{v} dV.$$

Because the map $u \mapsto u(x)$ is a bounded linear functional on $b^2(\Omega)$ for each $x \in \Omega$, there exists a unique function $R_\Omega(x, \cdot) \in b^2(\Omega)$ such that

$$u(x) = \int_{\Omega} u(y) \overline{R_\Omega(x, y)} dV(y)$$

for every $u \in b^2(\Omega)$. The function R_Ω , which can be viewed as a function on $\Omega \times \Omega$, is called the *reproducing kernel* of Ω .

We now derive some basic properties of R_Ω ; these are analogous to properties of the zonal harmonics we studied in Chapter 5 (even the proofs are the same).

Suppose $u \in b^2(\Omega)$ is real valued. Then

$$\begin{aligned} 0 &= \operatorname{Im} u(x) \\ &= \operatorname{Im} \int_{\Omega} u(y) \overline{R_\Omega(x, y)} dV(y) \\ &= - \int_{\Omega} u(y) \operatorname{Im} R_\Omega(x, y) dV(y). \end{aligned}$$

Taking $u = \operatorname{Im} R_\Omega(x, \cdot)$, this shows that

$$\int_{\Omega} (\operatorname{Im} R_\Omega(x, y))^2 dV(y) = 0,$$

which implies $\operatorname{Im} R_\Omega \equiv 0$. We conclude that each R_Ω is real valued.

Consider now any orthonormal basis $u_1, u_2, \dots$ of $b^2(\Omega)$. (Recall that $L^2(\Omega, dV)$, and hence $b^2(\Omega)$, is separable.) By standard Hilbert space theory,

$$\begin{aligned} 8.4 \quad R_\Omega(x, \cdot) &= \sum_{k=1}^{\infty} \langle R_\Omega(x, \cdot), u_k \rangle u_k \\ &= \sum_{k=1}^{\infty} \overline{u_k(x)} u_k \end{aligned}$$

for each $x \in \Omega$, where the infinite sums converge in norm in $b^2(\Omega)$. Since point evaluation is a continuous linear functional on $b^2(\Omega)$, the equation above implies that

$$R_\Omega(x, y) = \sum_{k=1}^{\infty} \overline{u_k(x)} u_k(y)$$

for all $x, y \in \Omega$. Because R_Ω is real valued, the last sum is unchanged after complex conjugation, and we deduce that

$$R_\Omega(x, y) = R_\Omega(y, x)$$

for all $x, y \in \Omega$.

Note that

$$\begin{aligned} \|R_\Omega(x, \cdot)\|_2^2 &= \langle R_\Omega(x, \cdot), R_\Omega(x, \cdot) \rangle \\ &= R_\Omega(x, x) \end{aligned}$$

for every $x \in \Omega$.

Because $b^2(\Omega)$ is a closed subspace of the Hilbert space $L^2(\Omega, dV)$, there is a unique orthogonal projection Q of $L^2(\Omega, dV)$ onto $b^2(\Omega)$. For $u \in L^2(\Omega, dV)$ and $x \in \Omega$ we have

$$\begin{aligned} 8.5 \quad Q[u](x) &= \langle Q[u], R_\Omega(x, \cdot) \rangle \\ &= \langle u, R_\Omega(x, \cdot) \rangle \\ &= \int_\Omega u(y) R_\Omega(x, y) dV(y). \end{aligned}$$

The Reproducing Kernel for the Ball

In this section we will find an explicit formula for the reproducing kernel of the unit ball. The discussion in the last section suggests that we first obtain an orthonormal basis for $b^2(B)$ and then try to evaluate the infinite sum in 8.4. This approach is feasible when $n = 2$ (see Exercise 14 of this chapter), but unfortunately there appears to be no canonical choice for an orthonormal basis of $b^2(B)$ when $n > 2$.

We overcome this difficulty by recalling that the zonal harmonics introduced in Chapter 5 are reproducing kernels for the space $\mathcal{H}_m(\mathbf{R}^n)$. Thus for $u \in \mathcal{H}_m(\mathbf{R}^n)$,

$$8.6 \quad u(x) = \int_S u(\zeta) Z_m(x, \zeta) d\sigma(\zeta)$$

for each $x \in \mathbf{R}^n$ by 5.18. By using polar coordinates, we will obtain an analogue of 8.6 involving integration over B instead of S .

First we extend the zonal harmonic Z_m to a function on $\mathbf{R}^n \times \mathbf{R}^n$. We do this by making Z_m homogeneous in the second variable as well as in the first; in other words, we set

$$8.7 \quad Z_m(x, y) = |x|^m |y|^m Z_m(x/|x|, y/|y|).$$

(If either x or y is 0, we define $Z_m(x, y)$ to be 0 when $m > 0$; when $m = 0$, we define Z_0 to be identically 1.) With this definition, $Z_m(x, \cdot) \in \mathcal{H}_m(\mathbf{R}^n)$ for each $x \in \mathbf{R}^n$; also, $Z_m(x, y) = Z_m(y, x)$ for all $x, y \in \mathbf{R}^n$.

We now derive the analogue of 8.6 for integration over B : For every $u \in \mathcal{H}_m(\mathbf{R}^n)$, we have

$$\begin{aligned}
 \int_B u(y) Z_m(x, y) dV(y) &= nV(B) \int_0^1 r^{n-1} \int_S u(r\zeta) Z_m(x, r\zeta) d\sigma(\zeta) dr \\
 &= nV(B) \int_0^1 r^{n+2m-1} \int_S u(\zeta) Z_m(x, \zeta) d\sigma(\zeta) dr \\
 &= nV(B) u(x) \int_0^1 r^{n+2m-1} dr \\
 &= \frac{nV(B)}{n+2m} u(x)
 \end{aligned}$$

for each $x \in \mathbf{R}^n$. In other words, $u(x)$ equals the inner product of u with $(n+2m)Z_m(x, \cdot)/(nV(B))$ for every $u \in \mathcal{H}_m(\mathbf{R}^n)$.

Now, $\mathcal{H}_k(\mathbf{R}^n)$ is orthogonal to $\mathcal{H}_m(\mathbf{R}^n)$ in $b^2(B)$ if $k \neq m$ (verify using polar coordinates and 5.3). Thus if u is a harmonic polynomial of degree M , then $u(x)$ equals the inner product of u with $\sum_{m=0}^M (n+2m)Z_m(x, \cdot)/(nV(B))$. Taking $M = \infty$ in the last sum gives us a good candidate for the reproducing kernel of the ball; Theorem 8.9 will show that this is the right guess. The following lemma will be useful in proving this theorem.

8.8 Lemma: *The set of harmonic polynomials is dense in $b^2(B)$.*

PROOF: First note that if $u \in L^2(B, dV)$, then $u_r \rightarrow u$ in $L^2(B, dV)$ as $r \rightarrow 1$. (For $u \in C(\bar{B})$, use uniform continuity; the general result follows because $C(\bar{B})$ is dense in $L^2(B, dV)$.) Thus any $u \in b^2(B)$ can be approximated in $b^2(B)$ by functions harmonic on $\bar{B}$. But by 5.23, every function harmonic on $\bar{B}$ can be approximated uniformly on $\bar{B}$ —and hence in $L^2(B, dV)$ —by harmonic polynomials. $\square$

8.9 Theorem: *If $x, y \in B$ then*

$$8.10 \quad R_B(x, y) = \frac{1}{nV(B)} \sum_{m=0}^{\infty} (n + 2m) Z_m(x, y).$$

The series converges absolutely and uniformly on $K \times B$ for every compact $K \subset B$.

PROOF: For $x, y \in B \setminus \{0\}$ we have

$$\begin{aligned} |Z_m(x, y)| &= |x|^m |y|^m |Z_m(x/|x|, y/|y|)| \\ &\leq |x|^m |y|^m h_m, \end{aligned}$$

where h_m is the dimension of $\mathcal{H}_m(\mathbf{R}^n)$ as in Chapter 5. The preceding inequality and Exercise 7 in Chapter 5 show that the infinite series in 8.10 has the convergence properties claimed in the theorem. Thus if $F(x, y)$ denotes the right side of 8.10, then $F(x, \cdot)$ is a bounded harmonic function on B for each $x \in B$. In particular, $F(x, \cdot) \in b^2(B)$ for each $x \in B$.

Now fix $x \in B$. The discussion before the statement of Lemma 8.8 shows that $u(x) = \langle u, F(x, \cdot) \rangle$ whenever u is a harmonic polynomial. Because point evaluation is continuous on $b^2(B)$ and harmonic polynomials are dense in $b^2(B)$, we have $u(x) = \langle u, F(x, \cdot) \rangle$ for all $u \in b^2(B)$. Hence F is the reproducing kernel for the ball. $\square$

Our next goal is to evaluate explicitly the infinite sum in 8.10. The key is formula 5.22 for the Poisson kernel, which states that for $x \in B$ and $\zeta \in S$,

$$P(x, \zeta) = \sum_{m=0}^{\infty} Z_m(x, \zeta).$$

For $x, y \in B$, this implies

$$\begin{aligned} \sum_{m=0}^{\infty} Z_m(x, y) &= \sum_{m=0}^{\infty} Z_m(|y|x, y/|y|) \\ &= P(|y|x, y/|y|) \\ &= \frac{1 - |x|^2 |y|^2}{(1 - 2x \cdot y + |x|^2 |y|^2)^{n/2}}, \end{aligned}$$

assuming $y \neq 0$. These equations suggest that we enlarge the domain

of the Poisson kernel by defining

$$8.11 \quad P(x, y) = \frac{1 - |x|^2|y|^2}{(1 - 2x \cdot y + |x|^2|y|^2)^{n/2}}$$

for all $x, y \in B$. We then have $P(x, y) = P(y, x)$ for all $x, y \in B$. Note also that $P(x, y) = P(|x|y, x/|x|)$; thus $P(x, \cdot)$ extends harmonically to $\bar{B}$ for each $x \in B$.

Returning to 8.10, observe that

$$\begin{aligned} \sum_{m=0}^{\infty} 2m Z_m(x, y) &= \sum_{m=0}^{\infty} \frac{d}{dt} t^{2m} Z_m(x, y)|_{t=1} \\ &= \frac{d}{dt} \left(\sum_{m=0}^{\infty} t^{2m} Z_m(x, y) \right)|_{t=1} \\ &= \frac{d}{dt} \left(\sum_{m=0}^{\infty} Z_m(tx, ty) \right)|_{t=1} \\ &= \frac{d}{dt} P(tx, ty)|_{t=1}. \end{aligned}$$

Thus 8.10 implies the beautiful equation

$$8.12 \quad R_B(x, y) = \frac{nP(x, y) + \frac{d}{dt} P(tx, ty)|_{t=1}}{nV(B)}.$$

This simple representation gives us a formula in closed form for $R_B(x, y)$.

8.13 Theorem: *Let $x, y \in B$. Then*

$$R_B(x, y) = \frac{(n-4)|x|^4|y|^4 + (8x \cdot y - 2n - 4)|x|^2|y|^2 + n}{nV(B)(1 - 2x \cdot y + |x|^2|y|^2)^{1+n/2}}.$$

PROOF: Compute using 8.12 and 8.11. □

In the rest of this section Q will denote the orthogonal projection of $L^2(B, dV)$ onto $b^2(B)$. The following result should be compared to Theorem 5.19.

8.14 Theorem: *Let u be a polynomial on $\mathbf{R}^n$ of degree m . Then $Q[u]$ is a polynomial of degree at most m . Moreover,*

$$Q[u](x) = \frac{1}{nV(B)} \sum_{k=0}^m (n+2k) \int_B Z_k(x, y) u(y) dV(y)$$

for every $x \in B$.

PROOF: For fixed $r \in (0, 1)$, the function u_r is a polynomial of degree m . By Corollary 5.7 we can write it on S as a sum of spherical harmonics of degree at most m . Thus

$$\int_S Z_k(x, \zeta) u(r\zeta) d\sigma(\zeta) = 0$$

for all $k > m$. Hence

$$\begin{aligned} \int_B Z_k(x, y) u(y) dV(y) &= nV(B) \int_0^1 r^{n+k-1} \int_S Z_k(x, \zeta) u(r\zeta) d\sigma(\zeta) dr \\ &= 0 \end{aligned}$$

for all $k > m$. Combining this result with 8.10 and 8.5 gives the desired equation. $\square$

Theorem 8.14 provides the algorithm used by the software described in Appendix B for computing $Q[u]$ explicitly whenever u is a polynomial. For example, if $n = 5$ and $u(x) = x_1^2$, then

$$Q[u](x) = \frac{5 + 28x_1^2 - 7x_2^2 - 7x_3^2 - 7x_4^2 - 7x_5^2}{35}.$$

Let us make an observation in passing. We have seen that if f is a polynomial on $\mathbf{R}^n$, then $P[f]$ and $Q[f]$ are both polynomials. Each can be thought of as the solution to a certain minimization problem. The polynomial $P[f]$ minimizes

$$\|f - u\|_{L^\infty(S, d\sigma)},$$

while $Q[f]$ minimizes

$$\|f - u\|_{L^2(B, dV)},$$

where both minimums are taken over all functions u harmonic on $\bar{B}$.

Examples in $b^p(B)$

Because the Poisson integral is a linear isometry of $L^p(S)$ onto $h^p(B)$ ($p > 1$) and of $M(S)$ onto $h^1(B)$ (Theorem 6.12), we easily see that $h^p(B) \neq h^q(B)$ whenever $p \neq q$. We now prove the analogous result for the Bergman spaces of B . Suppose $1 \leq p < q < \infty$. Because B has finite volume measure, we clearly have $b^q(B) \subset b^p(B)$. Consider the identity map from $b^q(B)$ into $b^p(B)$. This map is linear and one-to-one, and it is a bounded map by Hölder's inequality. If this map were onto, then the inverse mapping would be continuous by the open mapping theorem, and so there would exist a constant $C < \infty$ such that

$$8.15 \quad \|u\|_{b^q} \leq C \|u\|_{b^p}$$

for all $u \in b^p(B)$. We will show that 8.15 fails. For $m = 1, 2, \dots$, choose a homogeneous harmonic polynomial u_m of degree m , with $u_m \not\equiv 0$. Integrating in polar coordinates, we find

$$\|u_m\|_{b^p} = \left(\int_S |u_m|^p d\sigma \right)^{1/p} \left(nV(B) \int_0^1 r^{pm+n-1} dr \right)^{1/p},$$

a similar result holding for $\|u_m\|_{b^q}$. Because L^r -norms on S with respect to σ increase as r increases (Hölder's inequality), we have

$$\frac{\|u_m\|_{b^q}}{\|u_m\|_{b^p}} \geq \frac{(nV(B)/(qm+n))^{1/q}}{(nV(B)/(pm+n))^{1/p}}.$$

As $m \rightarrow \infty$, the expression on the right of the last inequality tends to ∞ . Therefore 8.15 fails, proving that the identity map from $b^q(B)$ into $b^p(B)$ is not onto. Thus $b^q(B)$ is properly contained in $b^p(B)$, as claimed.

We turn now to some other properties of the Bergman spaces on the ball. First note that $h^p(B) \subset b^p(B)$ for all $p \in [1, \infty)$, as an easy integration in polar coordinates shows. (In fact, $h^p(B) \subset b^q(B)$ for all $q < pm/(n-1)$; see Exercise 18 of this chapter.) However, each of the spaces $b^p(B)$ contains functions not belonging to any $h^q(B)$, as we show below. In fact, we will construct a function in every $b^p(B)$ that at every point of the unit sphere fails to have a radial limit; such a function cannot belong to any $h^q(B)$ by Corollary 6.36. We begin with a lemma that will be useful in this construction.

8.16 Lemma: *Let $f_m(\zeta) = e^{im\zeta_1}$ for $\zeta \in S$ and $m = 1, 2, \dots$. Then $P[f_m] \rightarrow 0$ uniformly on compact subsets of B as $m \rightarrow \infty$.*

PROOF: Let $g \in C(S)$. Using A.6 in Appendix A, we see that $\int_S f_m g \, d\sigma$ equals a constant (depending only on n) times

$$\int_{-1}^1 (1-t^2)^{\frac{n-3}{2}} e^{imt} \int_{S_{n-1}} g(t, \sqrt{1-t^2} \zeta) \, d\sigma_{n-1}(\zeta) \, dt,$$

where S_{n-1} denotes the unit sphere in $\mathbf{R}^{n-1}$ and $d\sigma_{n-1}$ denotes normalized surface area measure on S_{n-1} . The Riemann-Lebesgue Lemma then shows that $\int_S f_m g \, d\sigma \rightarrow 0$ as $m \rightarrow \infty$. In particular, taking $g = P(x, \cdot)$ for $x \in B$, we see that $P[f_m] \rightarrow 0$ pointwise on B .

Because $|f_m| \equiv 1$ on S , we have $|P[f_m]| \leq 1$ on B for each m . Thus by Theorem 2.6, every subsequence of $(P[f_m])$ contains a subsequence converging uniformly on compact subsets of B . Because we already know that $P[f_m] \rightarrow 0$ pointwise on B , we must have $P[f_m] \rightarrow 0$ uniformly on compact subsets of B . $\square$

The harmonic functions of Lemma 8.16 extend continuously to $\bar{B}$ with boundary values of modulus one everywhere on S , yet converge uniformly to zero on compact subsets of B . In this they resemble the harmonic functions z^m in the unit disk of the complex plane.

8.17 Theorem: *Let $\alpha: [0, 1) \rightarrow [1, \infty)$ be an increasing function with $\alpha(r) \rightarrow \infty$ as $r \rightarrow 1$. Then there exists a harmonic function u on B such that*

- (a) $|u(r\zeta)| < \alpha(r)$ for all $r \in [0, 1)$ and all $\zeta \in S$;
- (b) at every point of S , u fails to have a finite radial limit.

PROOF: Choose an increasing sequence $s_m \in [0, 1)$ such that $\alpha(s_m) > m + 1$. From the sequence $(P[f_m])$ of Lemma 8.16, choose a subsequence (v_m) with $|v_m| < 2^{-m}$ on $s_m B$. Suppose $r \in [s_m, s_{m+1})$. Because each v_m is bounded by 1 on B , we have

$$\begin{aligned}
\sum_{k=1}^{\infty} |v_k(r\zeta)| &= \sum_{k=1}^m |v_k(r\zeta)| + \sum_{k=m+1}^{\infty} |v_k(r\zeta)| \\
&< m + 2^{-(m+1)} + 2^{-(m+2)} + \dots \\
&< m + 1 \\
&< \alpha(s_m) \\
&\leq \alpha(r).
\end{aligned}$$

Thus $\sum |v_m(r\zeta)| < \alpha(r)$ for all $r \in (0, 1)$ and all $\zeta \in S$; furthermore, $\sum |v_m|$ converges uniformly on compact subsets of B .

From the sequence (v_m) we inductively extract a subsequence (u_m) in the following manner. Set $u_1 = v_1$. Because u_1 is continuous on $\bar{B}$, we may choose $r_1 \in [0, 1)$ such that $|u_1(r\zeta) - u_1(\zeta)| < 1/4$ for all $r \in [r_1, 1]$ and all $\zeta \in S$. Suppose we have chosen $u_1, u_2, \dots, u_m$ from (v_m) and radii $0 < r_1 < \dots < r_m < 1$ such that

$$8.18 \quad \sum_{k=1}^m |u_k(r\zeta) - u_k(s\zeta)| < 1/4 \quad \text{for all } r, s \in [r_m, 1], \zeta \in S.$$

We then select u_{m+1} such that $|u_{m+1}| < 2^{-(m+1)}$ on $r_m \bar{B}$. Now choose $r_{m+1} \in (r_m, 1)$ so that 8.18 holds with $m+1$ in place of m . The radius r_{m+1} can be chosen since each u_k is continuous on $\bar{B}$.

Having obtained the subsequence (u_m) from (v_m) (as well as the accompanying sequence (r_m) of radii), we define

$$u = \sum_{m=1}^{\infty} u_m.$$

From the first paragraph of the proof we know that $|u(r\zeta)| < \alpha(r)$ for all $r \in [0, 1)$, and that $\sum u_m$ converges uniformly on compact subsets of B , which implies that u is harmonic on B .

We now show that at each point of S , u fails to have a radial limit. (Here is where we use the fact that $|u_m| \equiv 1$ on S for every m .) We have

$$\begin{aligned}
|u(r_{m+1}\zeta) - u(r_m\zeta)| &\geq |u_{m+1}(r_{m+1}\zeta) - u_{m+1}(r_m\zeta)| \\
&\quad - \sum_{k \neq m+1} |u_k(r_{m+1}\zeta) - u_k(r_m\zeta)| \\
&\geq |u_{m+1}(\zeta)| - |u_{m+1}(r_{m+1}\zeta) - u_{m+1}(\zeta)| \\
&\quad - |u_{m+1}(r_m\zeta)| - 1/4 - 2 \sum_{m+2}^{\infty} 2^{-k} \\
&\geq 1 - 1/4 - 2^{-(m+1)} - 1/4 - 2 \sum_{m+2}^{\infty} 2^{-k} \\
&\geq 1/2 - 2 \sum_{m+1}^{\infty} 2^{-k}.
\end{aligned}$$

Thus for each $\zeta \in S$, the sequence $(u(r_m\zeta))$ fails to have a finite limit as $m \rightarrow \infty$, which implies that u fails to have a finite radial limit at ζ . $\square$

8.19 Corollary: *There is a function u belonging to $\bigcap_{p < \infty} b^p(B)$ such that at every point of S , u fails to have a radial limit.*

PROOF: Let $\alpha(r) = 1 + \log 1/(1-r)$, and let u be the corresponding function guaranteed by Theorem 8.17. Integrating in polar coordinates, we easily check that u belongs to $b^p(B)$ for every $p \in [1, \infty)$. $\square$

The Upper Half-Space

The goal of this section is to find an explicit formula for the reproducing kernel of the upper half-space. A well-motivated, although computationally tedious, method of deriving this formula is given in Exercise 20 of this chapter. We will present a slicker method relying on the magic of integration by parts.

As we did for B , we will derive the reproducing kernel for H in terms of the Poisson kernel. Recall that for $z \in H$ and $t \in \mathbf{R}^{n-1}$, the Poisson kernel for H is the function

$$P_H(z, t) = \frac{2}{nV(B)} \frac{z_n}{|z - t|^n}.$$

For $w \in \mathbf{R}^n$, define $\bar{w} = (w_1, \dots, w_{n-1}, -w_n)$; note that $\bar{w}$ is the usual complex conjugate of w on $\mathbf{R}^2 = \mathbf{C}$. We now extend the domain of P_H by defining

$$8.20 \quad P_H(z, w) = \frac{2}{nV(B)} \frac{z_n + w_n}{|z - \bar{w}|^n}$$

for $z \neq \bar{w}$. Note that $P_H(z, w) = P_H(w, z)$ and $P_H(z + (0, r), w) = P_H(z, w + (0, r))$ for $r \in \mathbf{R}$ whenever these expressions make sense. Thus

$$\begin{aligned} P_H(z, w) &= P_H(w, z) \\ &= P_H(w + (0, z_n), z - (0, z_n)) \end{aligned}$$

for all $z, w \in H$, which implies $P_H(z, \cdot)$ is harmonic on $\{w \in \mathbf{R}^n : w_n > -z_n\}$ for each $z \in H$ (being the translate of a harmonic function).

Before proving the main result of this section, Theorem 8.22, we prove an analogue of Lemma 8.8 for H .

8.21 Lemma: *The set of functions that are harmonic and square integrable on a half-space larger than H is dense in $b^2(H)$.*

PROOF: Let $u \in b^2(H)$. For $\delta > 0$, the function sending z to $u(z + (0, \delta))$ belongs to $b^2(\{z \in \mathbf{R}^n : z_n > -\delta\})$. But the functions $u(z + (0, \delta))$ converge to $u(z)$ in $L^2(H, dV)$ as $\delta \rightarrow 0$. (This follows by uniform continuity if u is continuous and has compact support in H ; the set of such functions is dense in $L^2(H, dV)$.) $\square$

8.22 Theorem: *For all $z, w \in H$,*

$$R_H(z, w) = -2 \frac{\partial}{\partial w_n} P_H(z, w) = \frac{4}{nV(B)} \frac{n(z_n + w_n)^2 - |z - \bar{w}|^2}{|z - \bar{w}|^{n+2}}.$$

PROOF: The second equality is a simple computation starting from 8.20. Note that this equality implies $\partial P_H(z, w)/\partial w_n$ belongs to $b^2(H)$ for each fixed $z \in H$ (see Exercise 1 in Appendix A). The remainder of the proof will be devoted to showing that the first equality holds.

Fix $z \in H$. Suppose $\delta > 0$ and $u \in b^2(\{w \in \mathbf{R}^n : w_n > -\delta\})$. Then

$$\begin{aligned}
8.23 \quad & \int_H u(w) \frac{\partial}{\partial w_n} P_H(z, w) dV(w) \\
&= \int_{\mathbf{R}^{n-1}} \int_0^\infty u(x, y) \frac{\partial}{\partial y} P_H(z, (x, y)) dy dx.
\end{aligned}$$

Now, u is bounded and harmonic on $\bar{H}$ by 8.1. Thus, after integrating by parts in the inner integral, the right side of 8.23 becomes

$$\begin{aligned}
& - \int_{\mathbf{R}^{n-1}} u(x, 0) P_H(z, x) dx \\
& - \int_{\mathbf{R}^{n-1}} \int_0^\infty \left[\frac{\partial}{\partial y} u(x, y) \right] P_H(z, (x, y)) dx dy,
\end{aligned}$$

which equals

$$8.24 \quad -u(z) - \int_0^\infty \int_{\mathbf{R}^{n-1}} \left[\frac{\partial}{\partial y} u(x, y) \right] P_H(z, (x, y)) dx dy.$$

Notice that we reversed the order of integration to arrive at 8.24. This is permissible if the integrand in 8.24 is integrable over H . To verify this, note that by Corollary 8.2 there exists a constant $C < \infty$ such that

$$\left| \frac{\partial}{\partial y} u(x, y) \right| \leq \frac{C}{(y + \delta)^{1+n/2}}.$$

Note also that

$$8.25 \quad P_H(z, (x, y)) = P_H(z + (0, y), (x, 0)),$$

which implies $\int_{\mathbf{R}^{n-1}} P_H(z, (x, y)) dx = 1$ for each $y > 0$. The reader can now easily verify that the integrand in 8.24 is integrable over H .

For each $y > 0$, the term in brackets in 8.24 is the restriction to $\mathbf{R}^{n-1}$ of the function $w \mapsto D_n u(w + (0, y))$, which is bounded and harmonic on $\bar{H}$. Thus by 8.25, the integral over $\mathbf{R}^{n-1}$ in 8.24 equals $(D_n u)(z + (0, 2y))$. Therefore 8.24 equals

$$-u(z) - \int_0^\infty (D_n u)(z + (0, 2y)) dy = -u(z)/2,$$

where the last equality holds because $u(z + (0, 2y)) \rightarrow 0$ as $y \rightarrow \infty$ (by 8.1).

Let $F(w) = -2\partial P_H(z, w)/\partial w_n$. We have shown that $u(z) = \langle u, F \rangle$ whenever u is harmonic and square integrable on a half-space larger than H . Because the set of such functions u is dense in $b^2(H)$ (Lemma 8.21), the proof is complete. $\square$

Exercises

1. Prove that $b^p(\mathbf{R}^n) = \{0\}$.
2. Suppose that $u \in b^p(\Omega)$. Prove that $d(x, \partial\Omega)^{n/p}|u(x)| \rightarrow 0$ as $x \rightarrow \partial\Omega$.
3. Prove that if $u \in b^p(\{x \in \mathbf{R}^n : |x| > 1\})$, then u is harmonic at ∞ .
4. Suppose $u \in b^p(H)$ and $y > 0$. Prove that $u(x, y) \rightarrow 0$ as $|x| \rightarrow \infty$ in $\mathbf{R}^{n-1}$.
5. Prove that if u is a harmonic function on $\mathbf{R}^n$ such that

$$\int_{\mathbf{R}^n} |u(x)|(1 + |x|)^\lambda dV(x) < \infty$$

for some $\lambda \in \mathbf{R}$, then u is a polynomial.

6. (a) Assume $n > 2$ and $p \geq n/(n-2)$. Prove that if u is in $b^p(B \setminus \{0\})$, then u has a removable singularity at 0.
 (b) Show that the constant $n/(n-2)$ in part (a) is sharp.
 (c) Show that there exists a function in $\bigcap_{p < \infty} b^p(B_2 \setminus \{0\})$ that fails to have a removable singularity at 0.
7. Prove that $b^p(\mathbf{R}^n \setminus \{0\}) = \{0\}$.
8. Prove that

$$R_{r\Omega+a}(x, y) = r^{-n} R_\Omega\left(\frac{x-a}{r}, \frac{y-a}{r}\right)$$

for all $r > 0$, $a \in \mathbf{R}^n$.

9. Prove that $\|R_\Omega(x, \cdot)\|_{b^2(\Omega)} \leq (V(B)d(x, \partial\Omega)^n)^{-1/2}$.

10. Suppose $\Omega_1 \subset \Omega_2 \subset \dots$ is an increasing sequence of open subsets of $\mathbf{R}^n$ and $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$. Prove that

$$R_{\Omega}(x, y) = \lim_{k \rightarrow \infty} R_{\Omega_k}(x, y)$$

for all $x, y \in \Omega$.

11. Suppose $a_1, \dots, a_m$ are points in Ω . Let A be the m -by- m matrix whose entry in row j , column k , equals $R_{\Omega}(a_j, a_k)$. Prove that A is positive semidefinite.
12. Show that the harmonic Bloch space is contained in $b^p(B)$ for every $p < \infty$. (See Exercise 11, Chapter 2, for the definition of the harmonic Bloch space.)
13. Show that $u(x) = \int_B u R_B(x, \cdot) dV$ for all $u \in b^p(B)$ and for all $p \in [1, \infty)$.
14. Assume $n = 2$, and set $u_k(re^{i\theta}) = r^{|k|}e^{ik\theta}$, $k = 0, \pm 1, \dots$. Find constants c_k so that $\{c_k u_k\}$ is an orthonormal basis for $b^2(B)$, and then use 8.4 to derive the reproducing kernel for B_2 .
15. (a) Prove there are positive constants C_1, C_2 such that

$$\frac{C_1}{(1 - |x|)^{n/2}} \leq \|R_B(x, \cdot)\|_2 \leq \frac{C_2}{(1 - |x|)^{n/2}}$$

for all $x \in B$.

(b) Find an estimate analogous to (a) for $\|R_H(z, \cdot)\|_2$.

16. Prove that $R_B(x, \cdot)/\|R_B(x, \cdot)\|_{b^2(B)}$ converges to 0 weak* in $b^2(B)$ as $|x| \rightarrow 1$.
17. Show that for fixed $\zeta \in S$, $P(\cdot, \zeta) \in b^p(B)$ for $p < n/(n-1)$. Also show that $P(\cdot, \zeta) \notin b^{n/(n-1)}(B)$.
18. Show that $h^p(B) \subset b^q(B)$ for $q < pn/(n-1)$.
19. Show that functions in $b^p(B)$ belong to h^q of balls internally tangent to B . More precisely, suppose $1 \leq q < (n-1)p/(2n)$ and $u \in b^p(B)$. Prove that if $a \in B \setminus \{0\}$, then the function $x \mapsto u(a + (1 - |a|)x)$ is in $h^q(B)$.

20. Derive the formula for R_H in Theorem 8.22 by writing $H = \bigcup_{k=1}^{\infty} B(k\mathbf{N}, k)$ and then using Exercise 10 and 8.13.
21. Show that every positive harmonic function on B belongs to $b^1(B)$. Are there any positive harmonic functions on H that belong to $b^1(H)$?
22. Suppose $\delta > 0$ and $u \in b^p(\{z \in \mathbf{R}^n : z_n > -\delta\})$. Show that $D^\alpha u \in b^p(H)$ for every multi-index α .

CHAPTER 9

The Decomposition Theorem

If $K \subset \Omega$ is compact and u is harmonic on $\Omega \setminus K$, then u might be badly behaved near both ∂K and $\partial\Omega$; see, for example, Theorem 11.18. In this chapter we will see that u is the sum of two harmonic functions, one extending harmonically across ∂K , the other extending harmonically across $\partial\Omega$. More precisely, u has a decomposition of the form

$$u = v + w$$

on $\Omega \setminus K$, where v is harmonic on Ω and w is harmonic on $\mathbf{R}^n \setminus K$. Furthermore, there is a canonical choice for w that makes this decomposition unique.

This result, which we call the decomposition theorem, has many applications. In this chapter we will use it to give another proof of Bôcher's Theorem, to show that bounded harmonic functions extend harmonically across smooth sets of dimension $n - 2$, and to prove the logarithmic conjugation theorem. In Chapter 10, we will use the decomposition theorem to obtain a "Laurent" series expansion for harmonic functions on annular domains in $\mathbf{R}^n$.

The Fundamental Solution of the Laplacian

We have already seen how important the functions $|x|^{2-n}$ ($n > 2$) and $\log|x|$ ($n = 2$) are to harmonic function theory. Another illustration

of their importance is that they give rise to integral operators that invert the Laplacian; we will need these operators in the proof of the decomposition theorem.

The *support* of a function g on $\mathbf{R}^n$, denoted $\text{supp } g$, is the closure of the set $\{x \in \mathbf{R}^n : g(x) \neq 0\}$. We let $C_c^k = C_c^k(\mathbf{R}^n)$ denote the set of functions in $C^k(\mathbf{R}^n)$ that have compact support. We will frequently use the abbreviation dy for the usual volume measure $dV(y)$.

We now show how g can be reconstructed from Δg if $g \in C_c^2$.

9.1 Theorem ($n > 2$): If $g \in C_c^2$, then

$$g(x) = \frac{1}{(2-n)nV(B)} \int_{\mathbf{R}^n} |x-y|^{2-n} (\Delta g)(y) dy$$

for every $x \in \mathbf{R}^n$.

9.2 Theorem ($n = 2$): If $g \in C_c^2$, then

$$g(x) = \frac{1}{2\pi} \int_{\mathbf{R}^2} \log |x-y| (\Delta g)(y) dy$$

for every $x \in \mathbf{R}^2$.

PROOF: We present the proof for $n > 2$, leaving the minor modifications needed for $n = 2$ to the reader (Exercise 1 of this chapter). Note first that the function $|x|^{2-n}$ is locally integrable on $\mathbf{R}^n$ (use polar coordinates).

Fix $x \in \mathbf{R}^n$. Choose r large enough so that $B(0, r)$ contains both x and $\text{supp } g$. For small $\varepsilon > 0$, set $\Omega_\varepsilon = B(0, r) \setminus \bar{B}(x, \varepsilon)$. Because g is supported in $B(0, r)$,

$$\int_{\mathbf{R}^n} |x-y|^{2-n} (\Delta g)(y) dy = \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} |x-y|^{2-n} (\Delta g)(y) dy.$$

Now apply Green's identity

$$\int_{\Omega} (u \Delta g - g \Delta u) dV = \int_{\partial \Omega} (u D_{\mathbf{n}} g - g D_{\mathbf{n}} u) ds,$$

with $u(y) = |x-y|^{2-n}$ and $\Omega = \Omega_\varepsilon$. Since $g = 0$ near $\partial B(0, r)$, only the surface integral over $\partial B(x, \varepsilon)$ comes into play. Recalling that the

unnormalized surface area of S is $nV(B)$ (see A.2 in Appendix A), we calculate that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} |x - y|^{2-n} (\Delta g)(y) dy = (2 - n)nV(B)g(x),$$

which gives the desired conclusion. $\square$

For $x \in \mathbf{R}^n \setminus \{0\}$, set

$$F(x) = \begin{cases} [(2 - n)nV(B)]^{-1}|x|^{2-n} & \text{if } n > 2 \\ (2\pi)^{-1} \log |x| & \text{if } n = 2. \end{cases}$$

The function F is called the *fundamental solution* of the Laplacian; it serves as the kernel of an integral operator that inverts the Laplacian on C_c^2 . To see this, define

$$9.3 \quad (Tg)(x) = \int_{\mathbf{R}^n} F(x - y)g(y) dy = \int_{\mathbf{R}^n} F(y)g(x - y) dy$$

for $g \in C_c$. Now suppose $g \in C_c^2$. Then $T(\Delta g) = g$ by 9.1 or 9.2. On the other hand, differentiation under the integral sign on the right side of 9.3 shows that $\Delta(Tg) = T(\Delta g)$; applying 9.1 or 9.2 again, we see that $\Delta(Tg) = g$. Thus $T \circ \Delta = \Delta \circ T = I$, the identity operator, on the space C_c^2 ; in other words, $T = \Delta^{-1}$ on C_c^2 .

We can now solve the inhomogeneous equation

$$9.4 \quad \Delta u = g$$

for any $g \in C_c^2$; we simply take $u = Tg$. Equation 9.4 is often referred to as *Poisson's equation*.

Decomposition of Harmonic Functions

The reader is already familiar with a result from complex analysis that can be interpreted as a decomposition theorem. Suppose $0 < r < R < \infty$, $K = \bar{B}(0, r)$, and $\Omega = B(0, R)$. Assume f is holomorphic on the annulus $\Omega \setminus K$, and let $\sum_{-\infty}^{\infty} a_k z^k$ be the Laurent expansion of f in $\Omega \setminus K$. Setting $g(z) = \sum_0^{\infty} a_k z^k$ and $h(z) = \sum_{-\infty}^{-1} a_k z^k$, we see that $f = g + h$ on $\Omega \setminus K$, that g extends to be holomorphic on Ω , and that h extends to be holomorphic on

$(\mathbf{C} \cup \{\infty\}) \setminus K$. The Laurent series expansion therefore gives us a decomposition for holomorphic functions (in the special case of annular regions in $\mathbf{C}$). The decomposition theorem (9.6 and 9.7) is the analogous result for harmonic functions.

We will need a large supply of smooth functions in the proof of the decomposition theorem; the following lemma guarantees this supply.

9.5 Lemma: *Suppose $K \subset \Omega$ is compact. Then there exists $\varphi \in C_c^\infty(\mathbf{R}^n)$ such that $\varphi \equiv 1$ on K , $\text{supp } \varphi \subset \Omega$, and $0 \leq \varphi \leq 1$ on $\mathbf{R}^n$.*

PROOF: Define a C^∞ -function f on $\mathbf{R}$ by setting

$$f(t) = \begin{cases} e^{-1/t} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0, \end{cases}$$

and define a function $\psi \in C_c^\infty(\mathbf{R}^n)$ by setting $\psi(y) = cf(1 - 2|y|^2)$, where the constant c is chosen so that $\int_{\mathbf{R}^n} \psi(y) dy = 1$. Note that $\text{supp } \psi \subset B$.

For $r > 0$, let $\psi_r(y) = r^{-n}\psi(y/r)$. Observe that $\text{supp } \psi_r \subset rB$ and that $\int_{\mathbf{R}^n} \psi_r(y) dy = 1$.

Now set $r = d(K, \partial\Omega)/3$ and define $\omega = \{x \in \Omega : d(x, K) < r\}$. Finally, put

$$\varphi(x) = \int_{\omega} \psi_r(x - y) dy$$

for $x \in \mathbf{R}^n$. Differentiation under the integral sign above shows that $\varphi \in C^\infty$. Clearly $0 \leq \varphi \leq 1$ on $\mathbf{R}^n$. Because $\psi_r(x - y)$ is supported in $B(x, r)$, we have $\varphi(x) = 1$ whenever $x \in K$ and $\varphi(x) = 0$ whenever $d(x, K) > 2r$. $\square$

We now prove the decomposition theorem; the $n > 2$ case differs from the $n = 2$ case, so we state the two results separately.

9.6 The Decomposition Theorem ($n > 2$): *Let K be a compact subset of Ω . If u is harmonic on $\Omega \setminus K$, then u has a unique decomposition of the form*

$$u = v + w,$$

where v is harmonic on Ω and w is a harmonic function on $\mathbf{R}^n \setminus K$ satisfying $\lim_{x \rightarrow \infty} w(x) = 0$.

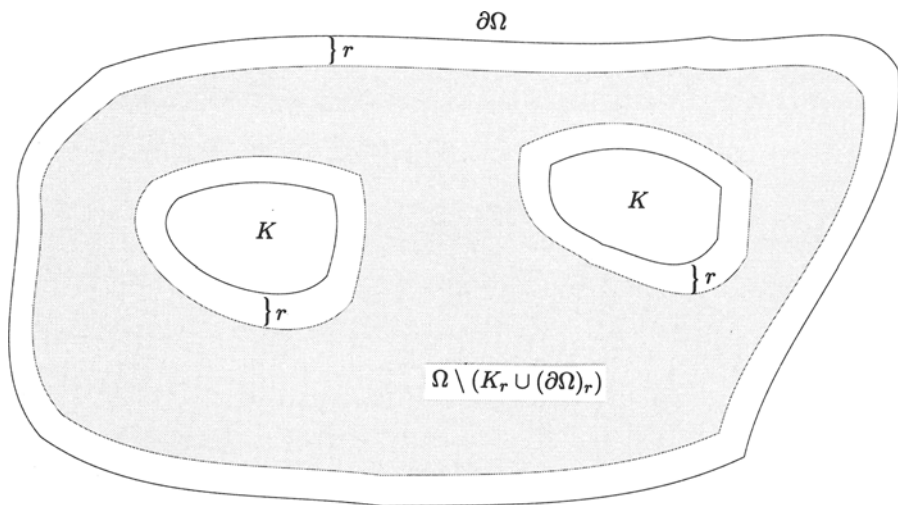
9.7 The Decomposition Theorem ($n = 2$): Let K be a compact subset of Ω . If u is harmonic on $\Omega \setminus K$, then u has a unique decomposition of the form

$$u = v + w,$$

where v is harmonic on Ω and w is a harmonic function on $\mathbf{R}^2 \setminus K$ satisfying $\lim_{x \rightarrow \infty} w(x) - c \log |x| = 0$ for some constant c .

PROOF: We present the proof for $n > 2$, leaving the changes needed for $n = 2$ to the reader (Exercise 3 of this chapter).

Suppose first that Ω is bounded. For $E \subset \mathbf{R}^n$ and $r > 0$, let $E_r = \{x \in \mathbf{R}^n : d(x, E) < r\}$. Choose r small enough so that K_r and $(\partial\Omega)_r$ are disjoint. By Lemma 9.5, there is a function $\varphi_r \in C_c^\infty(\mathbf{R}^n)$ supported in $\Omega \setminus K$ such that $\varphi_r \equiv 1$ on $\Omega \setminus (K_r \cup (\partial\Omega)_r)$; the following diagram may be helpful.



$\varphi_r \equiv 1$ on the shaded region.

For $x \in \Omega \setminus (K_r \cup (\partial\Omega)_r)$, apply Theorem 9.1 to the function $u\varphi_r$, which can be thought of as a function in $C_c^\infty(\mathbf{R}^n)$, to obtain

$$\begin{aligned}
u(x) &= (u\varphi_r)(x) \\
&= \frac{1}{(2-n)nV(B)} \int_{\mathbf{R}^n} |x-y|^{2-n} \Delta(u\varphi_r)(y) dy \\
&= \frac{1}{(2-n)nV(B)} \int_{(\partial\Omega)_r} |x-y|^{2-n} \Delta(u\varphi_r)(y) dy \\
&\quad + \frac{1}{(2-n)nV(B)} \int_{K_r} |x-y|^{2-n} \Delta(u\varphi_r)(y) dy \\
&= v_r(x) + w_r(x),
\end{aligned}$$

where $v_r(x)$ is $[(2-n)nV(B)]^{-1}$ times the integral over $(\partial\Omega)_r$ and $w_r(x)$ is $[(2-n)nV(B)]^{-1}$ times the integral over K_r . Differentiation under the integral sign shows that v_r is harmonic on $\Omega \setminus (\partial\Omega)_r$ and that w_r is harmonic on $\mathbf{R}^n \setminus K_r$. We also see that $w_r(x) \rightarrow 0$ as $x \rightarrow \infty$.

Suppose now that $s < r$. Then, as in the previous paragraph, we obtain the decomposition $u = v_s + w_s$ on $\Omega \setminus (K_s \cup \partial\Omega_s)$. We claim that $v_r = v_s$ on $\Omega \setminus (\partial\Omega)_r$ and $w_r = w_s$ on $(\mathbf{R}^n \cup \{\infty\}) \setminus K_r$. To see this, note that if $x \in \Omega \setminus (K_r \cup (\partial\Omega)_r)$, then $v_r(x) + w_r(x) = v_s(x) + w_s(x)$ (because both sides equal $u(x)$). Thus, $w_r - w_s$ is a harmonic function on $\mathbf{R}^n \setminus K_r$ that extends to be harmonic on $\mathbf{R}^n$ ($w_r - w_s$ agrees with $v_s - v_r$ near K_r). Because both w_r and w_s tend to 0 at infinity, Liouville's Theorem (2.1) implies that $w_r - w_s \equiv 0$. Thus $w_r = w_s$ and $v_r = v_s$ on $\Omega \setminus (K_r \cup (\partial\Omega)_r)$, as claimed.

For $x \in \Omega$, we may thus set $v(x) = v_r(x)$ for all r small enough so that $x \in \Omega \setminus (\partial\Omega)_r$. Similarly, for $x \in \mathbf{R}^n \setminus K$, we set $w(x) = w_r(x)$ for small r . We have arrived at the desired decomposition $u = v + w$.

Now suppose that Ω is unbounded and u is harmonic on $\Omega \setminus K$. Choose R large enough so that $K \subset B(0, R)$ and let $\omega = \Omega \cap B(0, R)$. Observe that K is a compact subset of the bounded open set ω and that u is harmonic on $\omega \setminus K$. Applying the result just proved for bounded open sets, we have

$$u(x) = \tilde{v}(x) + w(x)$$

for $x \in \omega \setminus K$, where $\tilde{v}$ is harmonic on ω and w is a harmonic function on $\mathbf{R}^n \setminus K$ satisfying $\lim_{x \rightarrow \infty} w(x) = 0$. Notice that the difference $u - w$ is harmonic on $\Omega \setminus K$ and extends harmonically across K because it agrees with $\tilde{v}$ near K . Set $v = u - w$; the sum $v + w$ is then the desired decomposition of u .

Finally, the proof of the uniqueness of the decomposition is similar to the proof given above that $w_r = w_s$ and $v_r = v_s$ on $\Omega \setminus (K_r \cup (\partial\Omega)_r)$. $\square$

Note that by Theorem 4.5, the function w of Theorem 9.6 is harmonic at ∞ .

Bôcher's Theorem Revisited

The remainder of this chapter consists of applications of the decomposition theorem. We begin by using it to give another proof of Bôcher's Theorem (3.9). Here we deal only with $n > 2$; easy modifications will give the $n = 2$ case. Another proof for the $n = 2$ case may be based on the logarithmic conjugation theorem presented later in this chapter (see Exercise 6 of this chapter).

9.8 Bôcher's Theorem: *Assume $n > 2$. Let $a \in \Omega$. If u is harmonic on $\Omega \setminus \{a\}$ and u is positive near a , then*

$$u = v + c|x - a|^{2-n}$$

for some harmonic function v on Ω and some nonnegative constant c .

PROOF: Without loss of generality we can assume u is real valued and $a = 0$. By the decomposition theorem (9.6), we can write $u = v + w$, where v is harmonic on Ω , w is harmonic on $\mathbf{R}^n \setminus \{0\}$, and $w(x) \rightarrow 0$ as $x \rightarrow \infty$. We will complete the proof by showing that $w(x) = c|x|^{2-n}$ for some nonnegative constant c .

Because u is positive near 0 and v is bounded near 0, $w = u - v$ is bounded below near 0. Let $\varepsilon > 0$ and set $h(x) = w(x) + \varepsilon|x|^{2-n}$. Then $\lim_{x \rightarrow \infty} h(x) = 0$ and $\lim_{x \rightarrow 0} h(x) = \infty$, so the minimum principle (1.6) implies that $h > 0$ on $\mathbf{R}^n \setminus \{0\}$. Letting $\varepsilon \rightarrow 0$, we conclude that $w \geq 0$ on $\mathbf{R}^n \setminus \{0\}$.

Now take the Kelvin transform of w . Because w tends to zero at ∞ , $K[w]$ has a removable singularity at 0 by Theorem 4.5. Thus $K[w]$ extends to be nonnegative and harmonic on all of $\mathbf{R}^n$. By Liouville's Theorem for positive harmonic functions (3.1), $K[w] = c$ for some nonnegative constant c . We therefore have $w(x) = c|x|^{2-n}$, completing the proof. $\square$

Removable Sets for Bounded Harmonic Functions

Let $h^\infty(\Omega)$ denote the collection of bounded harmonic functions on Ω . We say that a compact set $K \subset \Omega$ is h^∞ -removable for Ω if every bounded harmonic function on $\Omega \setminus K$ extends to be harmonic on Ω . The following theorem shows that if K is h^∞ -removable for one Ω containing K , then K is h^∞ -removable for every Ω containing K . Note that by Liouville's Theorem, K is h^∞ -removable for $\mathbf{R}^n$ if and only if every bounded harmonic function on $\mathbf{R}^n \setminus K$ is constant.

9.9 Theorem: *If K is a compact subset of Ω , then K is h^∞ -removable for Ω if and only if K is h^∞ -removable for $\mathbf{R}^n$.*

PROOF: Obviously, if K is h^∞ -removable for Ω , then K is h^∞ -removable for $\mathbf{R}^n$.

To prove the converse, we use the decomposition theorem. The $n > 2$ case is easy. Suppose that K is h^∞ -removable for $\mathbf{R}^n$ and that u is bounded and harmonic on $\Omega \setminus K$. Let $v + w$ be the decomposition of u given by (9.6). Because $n > 2$, $w(x) \rightarrow 0$ as $x \rightarrow \infty$. The boundedness near K of $w = u - v$ thus shows that w is bounded and harmonic on $\mathbf{R}^n \setminus K$. By hypothesis, w extends to be harmonic on $\mathbf{R}^n$. Since v is harmonic on Ω , $u = v + w$ extends to be harmonic on Ω .

The $n = 2$ case is more difficult (a rare occurrence); this is because w need not have limit 0 at ∞ . We will show that if there is a bounded harmonic function on $\Omega \setminus K$ that does not extend to be harmonic on Ω , then there is a nonconstant bounded harmonic function on $\mathbf{R}^2 \setminus K$.

We may assume that each connected component of K is a point, in other words, that K is totally disconnected. Otherwise some component of K consists of more than one point. The Riemann Mapping Theorem then implies the existence of a holomorphic map of the Riemann sphere minus that component onto B_2 , giving us a nonconstant bounded harmonic function on $\mathbf{R}^2 \setminus K$, as desired.

Let u be a bounded harmonic function on $\Omega \setminus K$ that does not extend to be harmonic on Ω . Then there exist distinct points x and y in K such that u does not extend harmonically to any neighborhood of x nor to any neighborhood of y . (If only one such point in

K existed, then we would have found a nonremovable isolated singularity of a bounded harmonic function, contradicting Theorem 2.3.)

Having obtained x and y , observe that the total disconnectivity of K shows that there exist disjoint open sets Ω_x and Ω_y (open in $\mathbf{R}^2$), with $x \in \Omega_x$ and $y \in \Omega_y$, such that $K \subset \Omega_x \cup \Omega_y$.

Now, u is harmonic on $(\Omega \cap \Omega_x) \setminus (\Omega_x \cap K)$, so by Theorem 9.7 we have the decomposition $u = v_x + w_x$, where v_x is harmonic on $\Omega \cap \Omega_x$ and w_x is harmonic on $\mathbf{R}^2 \setminus (\Omega_x \cap K)$, with $\lim_{z \rightarrow \infty} w_x(z) - c_x \log |z| = 0$ for some constant c_x . We have a similar decomposition of u on $(\Omega \cap \Omega_y) \setminus (\Omega_y \cap K)$. Note that w_x is not constant, otherwise u would extend harmonically to a neighborhood of x . Note also that if c_x were 0, then w_x would be a nonconstant bounded harmonic function on $\mathbf{R}^2 \setminus K$, and we would be done; we may thus assume that c_x is nonzero.

Setting $h = w_y - (c_y/c_x)w_x$, we claim h is the desired non-constant bounded harmonic function on $\mathbf{R}^2 \setminus K$. To see this, note that both w_x and w_y are bounded near K , and $\lim_{z \rightarrow \infty} h(z) = 0$; this proves h is bounded and harmonic on $\mathbf{R}^2 \setminus K$. If h were constant, then w_y would extend harmonically to a neighborhood of y , which would mean u extends harmonically to a neighborhood of y , a contradiction. $\square$

As an aside, note that the analogue of Theorem 9.9 for positive harmonic functions fails when $n = 2$: The compact set $\{0\}$ is removable for positive harmonic functions on $\mathbf{R}^2 \setminus \{0\}$ (3.3), but $\{0\}$ is not removable for positive harmonic functions on $B_2 \setminus \{0\}$.

Recall that if $K \subset \Omega$ is a single point, then K is h^∞ -removable for Ω (Theorem 2.3). Our next theorem improves that result, stating (roughly) that if the dimension of K is less than or equal to $n - 2$, then K is h^∞ -removable.

9.10 Theorem: *If $1 \leq k \leq n - 2$ and $\Psi: \bar{B}_k \rightarrow \Omega$ is a C^1 -map, then $\Psi(\bar{B}_k)$ is h^∞ -removable for Ω .*

PROOF: By Theorem 9.9, we need only show that if u is bounded and harmonic on $\mathbf{R}^n \setminus \Psi(\bar{B}_k)$, then u is constant. Without loss of generality, we assume u is real valued. By Proposition 4.7, there is a constant L such that u has limit L at ∞ . Let $\varepsilon > 0$ and set

$$v(x) = u(x) + \varepsilon \int_{B_k} |x - \Psi(y)|^{2-n} dV_k(y)$$

for $x \in \mathbf{R}^n \setminus \Psi(\bar{B}_k)$. Note that v is harmonic on $\mathbf{R}^n \setminus \Psi(\bar{B}_k)$ and that v has limit L at ∞ . Suppose we know that

$$9.11 \quad \int_{B_k} |x - \Psi(y)|^{2-n} dV_k(y) \rightarrow \infty \quad \text{as } x \rightarrow \Psi(\bar{B}_k).$$

The boundedness of u then shows $v(x) \rightarrow \infty$ as $x \rightarrow \Psi(\bar{B}_k)$. By the minimum principle, $v \geq L$ on $\mathbf{R}^n \setminus \Psi(\bar{B}_k)$. Letting $\varepsilon \rightarrow 0$, we conclude that $u \geq L$ on $\mathbf{R}^n \setminus \Psi(\bar{B}_k)$. A similar argument then gives $u \leq L$ on $\mathbf{R}^n \setminus \Psi(\bar{B}_k)$, so that u is constant, as desired. In other words, to complete the proof, we need only show that 9.11 holds.

To prove 9.11, first suppose that $x \in \Psi(\bar{B}_k)$. Then $x = \Psi(z)$ for some $z \in \bar{B}_k$. Because Ψ has a continuous derivative, there is a constant $C \in (0, \infty)$ such that $|\Psi(z) - \Psi(y)| \leq C|z - y|$ for every $y \in \bar{B}_k$. Thus

$$\begin{aligned} \int_{B_k} |x - \Psi(y)|^{2-n} dV_k(y) &= \int_{B_k} |\Psi(z) - \Psi(y)|^{2-n} dV_k(y) \\ &\geq C^{2-n} \int_{B_k} |z - y|^{2-n} dV_k(y) \\ &= \infty, \end{aligned}$$

where the last equality comes from Exercise 5 of this chapter. The assertion in 9.11 now follows from Fatou's Lemma. $\square$

Note that $\bar{B}_1$ is the interval $[-1, 1]$. Thus, any smooth compact arc in Ω is h^∞ -removable for Ω provided $n > 2$. Exercise 12 in Chapter 4 shows that compact sets of dimension $n - 1$ are not h^∞ -removable.

The Logarithmic Conjugation Theorem

In this section, Ω will denote a connected open subset of $\mathbf{R}^2$. We say that Ω is *finitely connected* if $\mathbf{R}^2 \setminus \Omega$ has finitely many bounded components. Recall that Ω is *simply connected* if $\mathbf{R}^2 \setminus \Omega$ has no bounded components.

If u is the real part of a holomorphic function f on Ω , then the imaginary part of f is called a *harmonic conjugate* of u . When Ω is simply connected, a real-valued harmonic function on Ω always has a harmonic conjugate ([2], Chapter VIII, Theorem 3.2).

The following theorem has been called the logarithmic conjugation theorem because it shows that a real-valued harmonic function on a finitely connected domain has a harmonic conjugate provided that some logarithmic terms are subtracted.

9.12 Logarithmic Conjugation Theorem: *Let Ω be a finitely connected domain. Let $K_1, \dots, K_m$ be the bounded components of $\mathbf{R}^2 \setminus \Omega$, and let $a_j \in K_j$ for $j = 1, \dots, m$. If u is real valued and harmonic on Ω , then there exist f holomorphic on Ω and $c_1, \dots, c_m \in \mathbf{R}$ such that*

$$u(z) = \operatorname{Re} f(z) + c_1 \log |z - a_1| + \dots + c_m \log |z - a_m|$$

for all $z \in \Omega$.

PROOF: We prove the theorem by induction on m , the number of bounded components in the complement of Ω . If $m = 0$, then Ω is simply connected and $u = \operatorname{Re} f$ for some function f holomorphic on Ω .

Suppose now that $m > 0$, and that the theorem is true with $m - 1$ in place of m . With Ω as in the statement of the theorem, set $\Omega' = \Omega \cup K_m$, so that Ω' is a finitely connected domain whose complement has $m - 1$ bounded components. Because u is harmonic on $\Omega' \setminus K_m$, 9.7 gives the decomposition $u = v + w$, where v is harmonic on Ω' and w is harmonic on $\mathbf{R}^2 \setminus K_m$, with $\lim_{z \rightarrow \infty} w(z) - c \log |z| = 0$ for some constant c .

Because v satisfies the induction hypothesis, we will be done if we can show that

$$\mathbf{9.13} \quad w(z) = \operatorname{Re} g(z) + c \log |z - a_m|$$

for some function g holomorphic on $\mathbf{R}^2 \setminus K_m$.

To verify 9.13, set

$$h(z) = w(z) - c \log |z - a_m|$$

for $z \in \mathbf{R}^2 \setminus K_m$. We easily calculate that $h(z) \rightarrow 0$ as $z \rightarrow \infty$; thus h extends to be harmonic on $(\mathbf{C} \cup \{\infty\}) \setminus K_m$. Now, $(\mathbf{C} \cup \{\infty\}) \setminus K_m$ can be viewed as a simply connected region on the Riemann sphere. On such a region every real-valued harmonic function has a harmonic conjugate. This gives 9.13, and thus completes the proof of the theorem. $\square$

As an application of the logarithmic conjugation theorem, we present a series development for functions harmonic on annuli.

9.14 Theorem: *If u is real valued and harmonic on the annulus $A = \{z \in \mathbf{R}^2 : r_0 < |z| < r_1\}$, then u has a series development of the form*

$$9.15 \quad u(re^{i\theta}) = c \log r + \sum_{k=-\infty}^{\infty} (c_k r^k + \overline{c_{-k}} r^{-k}) e^{ik\theta}.$$

The series converges absolutely for each $re^{i\theta} \in A$ and uniformly on compact subsets of A .

PROOF: Apply the logarithmic conjugation theorem (9.12) with $\Omega = A$, $K_1 = \{z \in \mathbf{R}^2 : |z| \leq r_0\}$, and $a_1 = 0$, to get

$$u(z) = c \log |z| + \operatorname{Re} f(z)$$

for some holomorphic function f on A . On A , f has a Laurent series expansion

$$f(z) = \sum_{k=-\infty}^{\infty} c_k z^k$$

that converges absolutely and uniformly on compact subsets of A . Now,

$$u(z) = c \log |z| + \frac{f(z) + \overline{f(z)}}{2};$$

the series representation 9.15 for u is obtained by setting $z = re^{i\theta}$ and replacing f with its Laurent series. $\square$

Note that the series representation 9.15 gives us another proof that the averages of u over circles of radius r satisfy conclusion (a) of 3.10.

In Chapter 10 we consider the problem of obtaining an analogous series representation for functions harmonic on annular domains in $\mathbf{R}^n$. There, as one might expect, the decomposition theorem (9.6, 9.7) will play an important role.

Additional applications of the logarithmic conjugation theorem may be found in [1].

Exercises

1. Prove Theorem 9.2.
2. Given $g \in C_c^2$, show that there exists a unique $u \in C_c^2$ such that $\Delta u = g$.
3. Prove the decomposition theorem in the $n = 2$ case (9.7).
4. (a) For $\Omega \subset \mathbf{R}^2$, define the operator $\bar{\partial}$ on $C^1(\Omega)$ by

$$\bar{\partial} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Show that if $f \in C^1(\Omega)$, then f is holomorphic on Ω if and only if $\bar{\partial}f = 0$.

(b) Show that if $g \in C_c^1(\mathbf{R}^2)$, then

$$g(w) = -\frac{1}{\pi} \int_{\mathbf{R}^2} \frac{(\bar{\partial}g)(z)}{w - z} dV_2(z)$$

for all $w \in \mathbf{C}$. (*Hint:* Imitate the proof of 9.1, using Green's Theorem instead of Green's identity.)

- (c) Let $\Omega \subset \mathbf{C}$, let $K \subset \Omega$ be compact, and let f be holomorphic on $\Omega \setminus K$. Using (b) and an argument similar to the proof of the decomposition theorem, prove that f has a unique decomposition of the form $f = g + h$, where g is holomorphic on Ω and h is holomorphic on $\mathbf{C} \setminus K$, with $\lim_{z \rightarrow \infty} h(z) = 0$.
5. Let $z \in \bar{B}$ and let $c \in \mathbf{R}$. Prove that

$$\int_{B_n} |z - x|^c dV(x) = \infty$$

if and only if $c \leq -n$.

6. **Theorem:** Let $\Omega \subset \mathbf{R}^2$ be an open set containing the point a . If u is harmonic on $\Omega \setminus \{a\}$ and u is positive near a , then $u(x) = v(x) - c \log |x - a|$ on Ω , where v is harmonic on Ω and c is a nonnegative constant.

- (a) Prove this by modifying the proof of Theorem 9.8.
 (b) Prove this by using the logarithmic conjugation theorem (9.12).

7. Does the conclusion of Theorem 9.10 remain true if we merely assume that Ψ is continuous on $\bar{B}_k$?
8. Let $\Omega \subset \mathbf{R}^2$ be finitely connected, and let $K_1, K_2, \dots, K_m$ be the bounded components of $\mathbf{R}^2 \setminus \Omega$. Let $a_j, b_j \in K_j$. Suppose that u is real valued and harmonic on Ω . Prove that if f, g are holomorphic functions on Ω and $c_j, d_j \in \mathbf{R}$ satisfy

$$\begin{aligned} u(z) &= \operatorname{Re} f(z) + c_1 \log |z - a_1| + \cdots + c_m \log |z - a_m| \\ &= \operatorname{Re} g(z) + d_1 \log |z - b_1| + \cdots + d_m \log |z - b_m|, \end{aligned}$$

then $c_j = d_j$. How are f and g related?

9. Use the series representation 9.15 to show that the Dirichlet problem for an annulus in $\mathbf{R}^2$ is solvable. More precisely, show that if A is an annulus in $\mathbf{R}^2$ and f is continuous on ∂A , then there is a function u harmonic on A and continuous on $\bar{A}$ such that $u|_{\partial A} = f$.

CHAPTER 10

Annular Regions

An *annular region* is a set of the form $\{x \in \mathbf{R}^n : r_0 < |x| < r_1\}$; here $r_0 \in [0, \infty)$ and $r_1 \in (0, \infty]$. Thus an annular region is the region between two concentric spheres, or is a punctured ball, or is the complement of a closed ball, or is $\mathbf{R}^n \setminus \{0\}$.

Laurent Series

If u is harmonic on B , then 5.23 gives the expansion

$$u(x) = \sum_{m=0}^{\infty} p_m(x),$$

where p_m is a homogeneous harmonic polynomial of degree m and the series converges absolutely and uniformly on compact subsets of B . This expansion is reminiscent of the power series expansion for holomorphic functions. We now take up the analogous Laurent series development for harmonic functions on annular regions.

10.1 Laurent Series ($n > 2$): Suppose u is harmonic on an annular region A . Then there exist unique homogeneous harmonic polynomials p_m and q_m of degree m such that

$$u(x) = \sum_{m=0}^{\infty} p_m(x) + \sum_{m=0}^{\infty} \frac{q_m(x)}{|x|^{2m+n-2}}$$

on A . The convergence is absolute and uniform on compact subsets of A .

PROOF: Suppose that A has inner radius $r_0 \in [0, \infty)$ and outer radius $r_1 \in (0, \infty]$. By the decomposition theorem (9.6) we have $u = v + w$, where v is harmonic on $r_1 B$ and w is harmonic on $(\mathbf{R}^n \cup \{\infty\}) \setminus r_0 \bar{B}$. Because v is harmonic on the ball $r_1 B$, there are homogeneous harmonic polynomials p_m such that

$$10.2 \quad v(x) = \sum_{m=0}^{\infty} p_m(x)$$

on $r_1 B$. The Kelvin transform of w , $K[w]$, is harmonic on the ball $(1/r_0)B$, and so there are homogeneous harmonic polynomials q_m such that

$$K[w](x) = \sum_{m=0}^{\infty} q_m(x)$$

on $(1/r_0)B$. Applying the Kelvin transform to both sides of this equation, we have

$$10.3 \quad w(x) = \sum_{m=0}^{\infty} \frac{q_m(x)}{|x|^{2m+n-2}}$$

on $\mathbf{R}^n \setminus r_0 \bar{B}$. Combining the series expansions 10.2 and 10.3, we obtain the desired expansion for u on A . The series 10.2 and 10.3 converge absolutely and uniformly on compact subsets of A , and hence so does the Laurent series expansion of u . Uniqueness of the expansion follows from the uniqueness of the decomposition $u = v + w$ and of the series expansions 10.2 and 10.3. $\square$

The preceding proof does not quite work when $n = 2$ because the decomposition theorem takes a different form in that case (see 9.6, 9.7). Exercise 1 of this chapter develops the Laurent series expansion for harmonic functions when $n = 2$.

Isolated Singularities

Suppose $n > 2$, $a \in \Omega$, and u is harmonic on $\Omega \setminus \{a\}$. By Theorem 10.1, there are homogeneous harmonic polynomials p_m and q_m such that

$$u(x) = \sum_{m=0}^{\infty} p_m(x-a) + \sum_{m=0}^{\infty} \frac{q_m(x-a)}{|x-a|^{2m+n-2}}$$

for x in a deleted neighborhood of a . We call the function

$$10.4 \quad \sum_{m=0}^{\infty} \frac{q_m(x-a)}{|x-a|^{2m+n-2}}$$

the *principal part* of u at a and classify the singularity at a accordingly: u has a *removable singularity* at a if each term in the principal part is zero; u has a *pole* at a if the principal part is a finite sum of nonzero terms; u has an *essential singularity* at a if the principal part has infinitely many nonzero terms.

If u has a pole at a , with principal part given by 10.4, and M is the largest integer such that $q_M \neq 0$, then we say that the pole has order $M + n - 2$. For example, if α is a multi-index, then $D^\alpha |x|^{2-n}$ has a pole of order $|\alpha| + n - 2$ at 0. Theorem 10.5(b) below shows why the order of a pole has been defined in this manner. We call a pole of order $n - 2$ a *fundamental pole* (because the principal part is then a multiple of the fundamental solution defined in Chapter 9).

10.5 Theorem ($n > 2$): *If u is harmonic with an isolated singularity at a , then u has*

(a) *a removable singularity at a if and only if*

$$\lim_{x \rightarrow a} |x-a|^{n-2} |u(x)| = 0;$$

(b) *a pole at a of order $M + n - 2$ if and only if*

$$0 < \limsup_{x \rightarrow a} |x-a|^{M+n-2} |u(x)| < \infty;$$

(c) *an essential singularity at a if and only if*

$$\limsup_{x \rightarrow a} |x-a|^N |u(x)| = \infty$$

for every positive integer N .

PROOF: The proof of (a) follows from Exercise 2(a) in Chapter 2.

For the remainder of the proof, we assume u has principal part at a given by $w(x) = \sum_{m=0}^{\infty} q_m(x-a)/|x-a|^{2m+n-2}$.

To prove (b), first suppose that u has a pole at a of order $M+n-2$. Then the homogeneity of each q_m implies

$$\limsup_{x \rightarrow a} |x-a|^{M+n-2} |u(x)| = \sup_S |q_M|.$$

The right side of this equation is positive and finite, and hence so is the left side, proving one direction of (b).

Conversely, suppose $0 < \limsup_{x \rightarrow a} |x-a|^{M+n-2} |u(x)| < \infty$. Then there is a constant $C < \infty$ such that $|w(a+r\zeta)| \leq C/r^{M+n-2}$ for small $r > 0$ and $\zeta \in S$. Let j be an integer with $j > M$. Then

$$\begin{aligned} \frac{\int_S |q_j(\zeta)|^2 d\sigma(\zeta)}{r^{2j+2n-4}} &\leq \sum_{m=0}^{\infty} \frac{\int_S |q_m(\zeta)|^2 d\sigma(\zeta)}{r^{2m+2n-4}} \\ &= \int_S |w(a+r\zeta)|^2 d\sigma(\zeta) \\ &\leq \frac{C^2}{r^{2M+2n-4}} \end{aligned}$$

for small $r > 0$; here we have used the orthogonality of spherical harmonics of different degree (5.3). Letting $r \rightarrow 0$, we obtain $\int_S |q_j|^2 d\sigma = 0$, so that $q_j \equiv 0$. Thus u has a pole at a of order at most $M+n-2$. Because $\limsup_{x \rightarrow a} |x-a|^{M+n-2} |u(x)|$ is positive, the order of the pole is at least $M+n-2$, completing the proof of (b).

To prove (c), first suppose that $\limsup_{x \rightarrow a} |x-a|^N |u(x)| = \infty$ for every positive integer N . By (a) and (b), u can have neither a removable singularity nor a pole at a , and thus u has an essential singularity at a .

Conversely, suppose there is a positive integer N such that $\limsup_{x \rightarrow a} |x-a|^N |u(x)| < \infty$. By the argument used in proving (b), this implies that $q_j \equiv 0$ for all sufficiently large j . Thus u does not have an essential singularity at a , completing the proof of (c). $\square$

The analogue of the theorem above for $n = 2$, along with the appropriate definitions, is given in Exercise 2 of this chapter.

Recall that Picard's Theorem states if f is a holomorphic function with essential singularity at a , then f assumes all complex values, with one possible exception, infinitely often on every deleted neighborhood of a . Picard's Theorem has the following analogue for real-valued harmonic functions.

10.6 Theorem: *Let u be a real-valued harmonic function with either an essential singularity or a pole of order greater than $n - 2$ at $a \in \mathbf{R}^n$. Then u assumes every real value infinitely often near a .*

PROOF: By Bôcher's Theorem (3.9), u cannot be bounded above or below on any deleted neighborhood of a . Thus, for every small $r > 0$, the connected set $u(B(a, r) \setminus \{a\})$ must be all of $\mathbf{R}$. This implies that u assumes every real value infinitely often near a . $\square$

There is no analogue of Theorem 10.6 for complex-valued harmonic functions.

The Residue Theorem

Suppose $u \in C^2(\Omega)$. Then u is harmonic on Ω if and only if

$$\int_{\partial B(a, r)} D_{\mathbf{n}} u \, ds = 0$$

for every closed ball $\bar{B}(a, r) \subset \Omega$; as usual, $D_{\mathbf{n}}$ denotes the derivative with respect to the outward normal $\mathbf{n}$ and ds denotes (unnormalized) surface measure. Proof: Apply Green's identity (take $v \equiv 1$) to small closed balls contained in Ω . We can think of this result as an analogue of Morera's Theorem for holomorphic functions.

Integrating the normal derivative over the boundary also yields a "residue theorem" of sorts. Suppose $n > 2$ and the harmonic function u has an isolated singularity at a , with Laurent series expansion at a given by

$$u(x) = \sum_{m=0}^{\infty} p_m(x-a) + \sum_{m=0}^{\infty} \frac{q_m(x-a)}{|x-a|^{2m+n-2}}.$$

We call the constant q_0 the *residue* of u at a , and write $\text{Res}(u, a) = q_0$. The following proposition and theorem justify this terminology.

10.7 Proposition ($n > 2$): *If u is harmonic on $\bar{B}(a, r) \setminus \{a\}$, then*

$$\text{Res}(u, a) = \frac{1}{(2-n)nV(B)} \int_{\partial B(a, r)} D_{\mathbf{n}} u \, ds.$$

PROOF: Without loss of generality, assume $a = 0$. Suppose

$$u(x) = \sum_{m=0}^{\infty} p_m(x) + \sum_{m=0}^{\infty} \frac{q_m(x)}{|x|^{2m+n-2}}$$

is the Laurent expansion of u about 0. The first sum is harmonic on $\bar{B}(0, r)$; hence, the integral of its normal derivative over $\partial B(0, r)$ is zero. The integral of the normal derivative of the second sum equals

$$q_0 r^{1-n} (2-n) \int_{\partial B(0, r)} ds + \sum_{m=1}^{\infty} (2-n-m) r^{1-n-2m} \int_{\partial B(0, r)} q_m \, ds.$$

The value of the first integral is $q_0 nV(B)(2-n)$; all other integrals vanish by the mean-value property. $\square$

10.8 Residue Theorem ($n > 2$): *Suppose Ω is a bounded open set with smooth boundary. Let $a_1, \dots, a_k$ be distinct points in Ω . If u is harmonic on $\bar{\Omega} \setminus \{a_1, \dots, a_k\}$, then*

$$\int_{\partial \Omega} D_{\mathbf{n}} u \, ds = (2-n)nV(B) \sum_{j=1}^k \text{Res}(u, a_j).$$

PROOF: Choose positive numbers $r_1, r_2, \dots, r_k$ small enough so that $\bar{B}(a_j, r_j) \subset \Omega$ for each j . Set $\omega = \Omega \setminus (\bigcup_{j=1}^k \bar{B}(a_j, r_j))$. Then

$$\int_{\partial \omega} D_{\mathbf{n}} u \, ds = 0$$

by Green's identity. Hence

$$\begin{aligned} \int_{\partial \Omega} D_{\mathbf{n}} u \, ds &= - \sum_{j=1}^k \int_{\partial B(a_j, r_j)} D_{\mathbf{n}} u \, ds \\ &= (2-n)nV(B) \sum_{j=1}^k \text{Res}(u, a_j) \end{aligned}$$

(note that $\mathbf{n}$ points toward a_j on $\partial B(a_j, r_j)$).

□

See Exercise 8 of this chapter for the analogous results when $n = 2$.

The Poisson Kernel for Annular Regions

Let A be a bounded annular region. If f is a continuous function on ∂A , does f have a continuous extension to $\bar{A}$ that is harmonic on A ? In this section we will see that this question, called the Dirichlet problem for A , has an affirmative answer if the inner radius of A is positive. In fact, we will find a Poisson-integral type formula for the solution. (In the next chapter, we show that the Dirichlet problem is solvable on a much wider class of domains, although in the more general context we will not have an explicit integral formula for the solution.)

Fix $r_0 \in (0, 1)$. Throughout this section, we assume that A is the annular region $\{x \in \mathbf{R}^n : r_0 < |x| < 1\}$. This is no loss of generality because dilations preserve harmonic functions.

To discover the formula for solving the Dirichlet problem on A , we begin with a special case. Suppose $g \in \mathcal{H}_m(S)$ for some $m \geq 0$. Consider the problem of finding a continuous function u on $\bar{A}$ that is harmonic on A , with $u = g$ on S and $u = 0$ on $r_0 S$. We first extend g to a harmonic homogeneous polynomial of degree m (which we also denote by g). The Kelvin transform of g is then harmonic on $\mathbf{R}^n \setminus \{0\}$; the homogeneity of g shows that $K[g](x) = g(x)/|x|^{2m+n-2}$. Thus the function u defined by

$$u(x) = \frac{1 - (r_0/|x|)^{2m+n-2}}{1 - r_0^{2m+n-2}} g(x)$$

solves the Dirichlet problem in this special case.

Let us define

$$10.9 \quad b_m(x) = \frac{1 - (r_0/|x|)^{2m+n-2}}{1 - r_0^{2m+n-2}},$$

so that $u(x) = b_m(x)g(x)$, where u is the function displayed above. Recall that integration against the zonal harmonic $Z_m(x, \zeta)$ reproduces the values of functions in $\mathcal{H}_m(\mathbf{R}^n)$ (5.18). Thus we can rewrite our formula for the solution u as follows:

$$u(x) = b_m(x) \int_S g(\zeta) Z_m(x, \zeta) d\sigma(\zeta).$$

Going one step further, if $g = \sum_{m=0}^M g_m$, where $g_m \in \mathcal{H}_m(S)$, then adding the solutions obtained for each g_m in the previous paragraph solves the Dirichlet problem for A with boundary data g on S and 0 on r_0S . Explicitly,

$$u(x) = \int_S g(\zeta) \left(\sum_{m=0}^M b_m(x) Z_m(x, \zeta) \right) d\sigma(\zeta).$$

Note that for each $\zeta \in S$, the function $x \mapsto b_m(x) Z_m(x, \zeta)$ is harmonic on A (because it equals a constant times $Z_m(x, \zeta)$ plus a constant times $K[Z_m(\cdot, \zeta)](x)$).

Now, any polynomial restricted to S is the sum of spherical harmonics (5.7). Furthermore, the set of polynomials is dense in $C(S)$ by the Stone-Weierstrass Theorem (see [8], Theorem 7.33). Suppose, then, that g is an arbitrary continuous function on S . To find a continuous function u on $\bar{A}$ that is harmonic on A , with $u = g$ on S and $u = 0$ on r_0S , the results above suggest that we try

$$u(x) = \int_S g(\zeta) P_A(x, \zeta) d\sigma(\zeta),$$

where

$$10.10 \quad P_A(x, \zeta) = \sum_{m=0}^{\infty} b_m(x) Z_m(x, \zeta).$$

Note that $0 < b_m(x) < 1$ for $x \in A$; thus, as in the proof of Theorem 5.21, the last series converges absolutely and uniformly on $K \times S$ for every compact $K \subset A$. In particular, for each $\zeta \in S$, the function $P_A(\cdot, \zeta)$ is harmonic on A .

We handle the Dirichlet problem for A with boundary data h on r_0S and 0 on S in a similar manner. Thus a process like the one above suggests that the solution u is given by the formula

$$u(x) = \int_S h(r_0\zeta) P_A(x, r_0\zeta) d\sigma(\zeta),$$

where

$$10.11 \quad P_A(x, r_0\zeta) = \sum_{m=0}^{\infty} c_m(x) Z_m(x, \zeta)$$

and

$$10.12 \quad c_m(x) = |x|^{-m} (r_0/|x|)^{m+n-2} \frac{1 - |x|^{2m+n-2}}{1 - r_0^{2m+n-2}}.$$

Note that $|c_m(x) Z_m(x, \zeta)| < (r_0/|x|)^{m+n-2} |Z_m(x/|x|, \zeta)|$ for $x \in A$, so the infinite sum in 10.11 converges absolutely and uniformly on $K \times S$ for every compact $K \subset A$, as in the proof of Theorem 5.21. In particular, for each $\zeta \in S$, the function $P_A(\cdot, r_0\zeta)$ is harmonic on A .

Having approached the Dirichlet problem for A “one sphere at a time”, we easily guess what to do for an arbitrary $f \in C(\partial A)$: We simply add the two candidate solutions obtained above.

We now make the formal definitions. For $n > 2$, P_A is the function on $A \times \partial A$ defined by 10.9–10.12. (For $n = 2$ and $m = 0$, the terms $b_0(x)$ and $c_0(x)$ must be replaced by multiples of $\log |x|$; Exercise 10 of this chapter asks the reader to make the necessary modifications.) For $f \in C(\partial A)$, the Poisson integral of f , denoted $P_A[f]$, is the function on A defined by

$$P_A[f](x) = \int_S f(\zeta) P_A(x, \zeta) d\sigma(\zeta) + \int_S f(r_0\zeta) P_A(x, r_0\zeta) d\sigma(\zeta).$$

We have already done most of the work needed to show that $P_A[f]$ solves the Dirichlet problem for f .

10.13 Theorem ($n > 2$): Suppose f is continuous on ∂A . Define u on $\bar{A}$ by

$$u(x) = \begin{cases} P_A[f](x) & \text{if } x \in B \\ f(x) & \text{if } x \in S. \end{cases}$$

Then u is continuous on $\bar{A}$ and harmonic on A .

PROOF: The function $P_A[f]$ is the sum of two harmonic functions, and hence is harmonic.

To complete the proof, we need only show that u is continuous on $\bar{A}$. The discussion above shows that u is continuous on $\bar{A}$ in the case where $f|_S$ and $f(r_0\cdot)|_S$ are both finite sums of spherical harmonics. By Corollary 5.7 and the Stone-Weierstrass Theorem

(see [8], Theorem 7.33), such functions are dense in $C(\partial A)$. For the general $f \in C(\partial A)$, we approximate f uniformly on ∂A with functions $f_1, f_2, \dots$ from this dense subspace; the corresponding solutions $u_1, u_2, \dots$ then converge uniformly to u on $\bar{A}$ by the maximum principle, proving that u is continuous on $\bar{A}$. $\square$

The hypothesis that r_0 be greater than 0 is needed to solve the Dirichlet problem for the annular region A . For example, there is no function u harmonic on the punctured ball $B \setminus \{0\}$, with u continuous on $\bar{B}$, satisfying $u = 1$ on S and $u(0) = 0$: If there were such a function, then it would be bounded and harmonic on $B \setminus \{0\}$, hence would extend harmonically to B (by 2.3), contradicting the minimum principle.

The results in this section are used by the software described in Appendix B to solve the Dirichlet problem for annular regions. For example, the software computes that the function harmonic on $\{x \in \mathbf{R}^3 : 2 \leq |x| \leq 3\}$ that equals x_1^2 when $|x| = 2$ and equals $x_1 x_2 x_3$ when $|x| = 3$ is

$$\begin{aligned} & -\frac{8}{3} + \frac{8}{|x|} - \frac{64x_1^2}{633} + \frac{5184x_1^2}{211|x|^5} + \frac{32x_2^2}{633} - \frac{2592x_2^2}{211|x|^5} + \frac{2187x_1x_2x_3}{2059} \\ & - \frac{279936x_1x_2x_3}{2059|x|^7} + \frac{32x_3^2}{633} - \frac{2592x_3^2}{211|x|^5}. \end{aligned}$$

Exercises

1. Suppose u is harmonic on an annular region A in $\mathbf{R}^2$. Show that there exist unique homogeneous harmonic polynomials p_m and q_m of degree m such that

$$u(x) = \sum_{m=0}^{\infty} p_m(x) + q_0 \log |x| + \sum_{m=1}^{\infty} \frac{q_m(x)}{|x|^{2m}}$$

on A . Show also that the series converges absolutely and uniformly on compact subsets of A .

2. Suppose u is a harmonic function with an isolated singularity at $a \in \mathbf{R}^2$. The *principal part* of u at a is defined to be

$$q_0 \log |x - a| + \sum_{m=1}^{\infty} \frac{q_m(x - a)}{|x - a|^{2m}},$$

where u has been expanded about a as in Exercise 1. We say that u has a *fundamental pole* at a if the principal part is a nonzero multiple of $\log |x|$. We say that u has a *pole* at a of order M if there is a largest positive integer M such that $q_M \neq 0$. We say that u has an *essential singularity* at a if the principal part has infinitely many nonzero terms. Prove that u has

- (a) a removable singularity at a if and only if

$$\lim_{x \rightarrow a} \frac{u(x)}{\log |x - a|} = 0;$$

- (b) a fundamental pole at a if and only if

$$0 < \lim_{x \rightarrow a} \left| \frac{u(x)}{\log |x - a|} \right| < \infty;$$

(c) a pole at a of order M if and only if

$$0 < \limsup_{x \rightarrow a} |x - a|^M |u(x)| < \infty;$$

(d) an essential singularity at a if and only if

$$\limsup_{x \rightarrow a} |x - a|^N |u(x)| = \infty$$

for every positive integer N .

3. Give an example of a harmonic function of n variables, $n > 2$, that has an essential singularity at 0.

4. Let u be a harmonic function with an isolated singularity at $a \in \mathbf{R}^n$. Show that u has a fundamental pole at a if and only if

$$\lim_{x \rightarrow a} |u(x)| = \infty.$$

5. Suppose $n > 2$ and u is a harmonic function with an isolated singularity at a . Prove that $\lim_{x \rightarrow a} |x - a|^{n-2} u(x)$ exists (as a complex number) if and only if u has either a removable singularity or a fundamental pole at a .

6. *Singularities at ∞* : Suppose u is harmonic on a deleted neighborhood of ∞ . The singularity of u at ∞ is classified using the Laurent expansion of $K[u]$ at 0; for example, if the Laurent expansion of $K[u]$ at 0 has vanishing principal part, then we say u has a *removable singularity* at ∞ .

(a) Show that u has an essential singularity at ∞ if and only if

$$\limsup_{x \rightarrow \infty} \frac{|u(x)|}{|x|^M} = \infty$$

for every positive integer M .

(b) Find growth estimates that characterize the other types of singularities at ∞ .

7. (a) Identify those functions that are harmonic on $\mathbf{R}^n$, $n > 2$, with fundamental pole at ∞ .

(b) Identify those functions that are harmonic on $\mathbf{R}^2$ with fundamental pole at ∞ .

8. Suppose $n = 2$ and the harmonic function u has an isolated singularity at $a \in \mathbf{R}^2$, with Laurent series expansion

$$u(x) = \sum_{m=0}^{\infty} p_m(x-a) + q_0 \log |x| + \sum_{m=1}^{\infty} \frac{q_m(x-a)}{|x-a|^{2m}}.$$

We say that the constant q_0 is the *residue* of u at a and write $\text{Res}(u, a) = q_0$. Prove that if u is harmonic on $\bar{B}(a, r) \setminus \{a\}$, then

$$\text{Res}(u, a) = \frac{1}{2\pi} \int_{\partial B(a, r)} D_{\mathbf{n}} u \, ds.$$

Also prove an analogue of the residue theorem (10.8) for the case $n = 2$.

9. Show how formulas 10.11 and 10.12 are derived.
10. Find the correct replacements for 10.9 and 10.12 when $n = 2$ and $m = 0$, and use this to solve the Dirichlet problem for annular regions in the plane.
11. Let $0 < r_0 < 1$ and let $A = \{x \in \mathbf{R}^n : r_0 < |x| < 1\}$. Let p, q be polynomials on $\mathbf{R}^n$, and let f be the function on ∂A that equals p on $r_0 S$ and equals q on S . Prove that $P_A[f]$ extends to a function that is harmonic on $\mathbf{R}^n \setminus \{0\}$.
12. *Generalized Annular Dirichlet Problem:* Suppose that $A = \{x \in \mathbf{R}^n : r_0 < |x| < r_1\}$, where $0 < r_0 < r_1 < \infty$, and f, g , and h are polynomials on $\mathbf{R}^n$. Prove there is a function $u \in C(\bar{A})$ such that $u = f$ on $r_0 S$, $u = g$ on $r_1 S$, and $\Delta u = h$ on A . Show that if $n > 2$, then u is a finite sum of functions of the form $p(x)/|x|^m$, where p is a polynomial on $\mathbf{R}^n$ and m is a nonnegative integer. (The software described in Appendix B can find u explicitly.)

CHAPTER 11

The Dirichlet Problem and Boundary Behavior

In this chapter we construct harmonic functions on Ω that behave in a prescribed manner near $\partial\Omega$. Here we are interested in general domains $\Omega \subset \mathbf{R}^n$; the techniques we developed for the special domains B and H will thus not be available. Most of this chapter will concern the Dirichlet problem. In the last section, however, we will study a different kind of boundary behavior problem—the construction of harmonic functions on Ω that cannot be extended harmonically across any part of $\partial\Omega$.

The Dirichlet Problem

If f is a continuous function on $\partial\Omega$, does f have a continuous extension to $\bar{\Omega}$ that is harmonic on Ω ? This is the *Dirichlet problem* for Ω with boundary data f . If the answer is affirmative for all continuous f on $\partial\Omega$, we say that the Dirichlet problem for Ω is *solvable*. Recall that the Dirichlet problem is solvable for B (Theorem 1.14) and for the region between two concentric spheres (Theorem 10.13), but not for the punctured ball $B \setminus \{0\}$ (see the remark following the proof of 10.13). The Dirichlet problem is sometimes referred to as “the first boundary value problem of potential theory”; the search for its solution led to the development of much of harmonic function theory.

We will obtain a necessary and sufficient condition for the Dirichlet problem to be solvable for bounded Ω . Although the condition is not entirely satisfactory, it leads in many cases to easily verified geometric criteria that imply the Dirichlet problem is solvable. We will see, for example, that the Dirichlet problem is solvable for bounded Ω whenever Ω is convex, or whenever $\partial\Omega$ is “smooth”.

Note that when Ω is bounded, the maximum principle (1.5) implies that if a solution to the Dirichlet problem exists, then it is unique.

Subharmonic Functions

We follow the so-called Perron approach in solving the Dirichlet problem. This ingenious method constructs a solution as the supremum of a family of subharmonic functions. In this book, we call a real-valued function u *subharmonic* on Ω provided u is continuous on Ω and u satisfies the *submean-value property* on Ω . The latter requirement is that for each $a \in \Omega$ there exists a closed ball $\bar{B}(a, R) \subset \Omega$ such that

$$11.1 \quad u(a) \leq \int_S u(a + r\zeta) d\sigma(\zeta)$$

whenever $0 < r \leq R$. Note that we are not requiring 11.1 to hold for all $r < d(a, \partial\Omega)$. (But see Exercise 4 of this chapter.)

A real-valued harmonic function on Ω is obviously subharmonic on Ω . A finite sum of subharmonic functions is subharmonic, as is any positive scalar multiple of a subharmonic function. In Exercise 7 of this chapter we ask the reader to prove that a real-valued $u \in C^2(\Omega)$ is subharmonic on Ω if and only if $\Delta u \geq 0$ on Ω . Thus $u(x) = |x|^2$ is a subharmonic function on $\mathbf{R}^n$ that is not harmonic. This example shows that subharmonic functions do not satisfy the minimum principle. They do, however, satisfy the maximum principle.

11.2 Theorem: *Suppose Ω is connected and u is subharmonic on Ω . If u has a maximum in Ω , then u is constant.*

PROOF: Suppose u attains a maximum at $a \in \Omega$. Choose a closed ball $\bar{B}(a, R) \subset \Omega$ as in 11.1. We have $u \leq u(a)$ in $B(a, R)$. If

u were less than $u(a)$ at any point of $B(a, R)$, then the continuity of u would show that 11.1 fails for some $r < R$. Thus $u \equiv u(a)$ on $B(a, R)$. The set where u attains its maximum is therefore an open subset of Ω . Because this set is also closed in Ω , it must be all of Ω by the connectivity of Ω , proving that u is constant on Ω . $\square$

The following theorem indicates another sense in which subharmonic functions are “sub”-harmonic.

11.3 Theorem: *Let Ω be bounded. Suppose u and v are continuous on $\bar{\Omega}$, u is subharmonic on Ω , and v is harmonic on Ω . If $u \leq v$ on $\partial\Omega$, then $u \leq v$ on $\bar{\Omega}$.*

PROOF: We have $u - v \leq 0$ on $\partial\Omega$. Because $u - v$ is subharmonic on Ω , the maximum principle (11.2) shows that $u - v \leq 0$ throughout Ω . $\square$

The proof of the next result follows easily from the definition of subharmonic functions, and we leave it to the reader.

11.4 Proposition: *If u_1 and u_2 are subharmonic on Ω , then $\max\{u_1, u_2\}$ is subharmonic on Ω .*

Although Proposition 11.4 is easy, it indicates why subharmonic functions are useful: They can be “bent” in ways that harmonic functions simply would not tolerate. Here is a more sophisticated bending result:

11.5 Theorem: *Suppose that u is subharmonic on Ω and $K \subset \Omega$ is a closed ball. Let v denote the continuous function on K that equals u on ∂K and is harmonic on the interior of K . Then the function w defined on Ω by*

$$w = \begin{cases} v & \text{on } K \\ u & \text{on } \Omega \setminus K \end{cases}$$

is subharmonic on Ω . Furthermore, $u \leq w$ on Ω .

PROOF: The continuity of w on Ω is clear, and the inequality $u \leq w$ on Ω follows from Theorem 11.3. To verify the submean-value property for w , let $a \in \Omega$. If a is in the interior of K , the harmonicity

of w near a gives equality in 11.1 for small r . If a is not in the interior of K , then $u(a) = w(a)$. The subharmonicity of u , coupled with the inequality $u \leq w$ on Ω , then shows that w satisfies 11.1 for small r . It follows that w is subharmonic on Ω . $\square$

We call the function w defined in Theorem 11.5 the *Poisson modification* of u for K .

The Perron Construction

In this and the next section, Ω will denote a bounded open subset of $\mathbf{R}^n$ and f will denote a continuous real-valued function on $\partial\Omega$. (Note that the Dirichlet problem is no less general if one assumes the boundary data to be real.) We define $\mathcal{S}_f$ to be the family of real-valued, continuous functions u on $\bar{\Omega}$ that are subharmonic on Ω and satisfy $u \leq f$ on $\partial\Omega$. The collection $\mathcal{S}_f$ is often called the *Perron family* for f . Note that $\mathcal{S}_f$ is never empty—it contains the constant function $u(x) = m$, where m is the minimum value of f on $\partial\Omega$. (We are already using the boundedness of Ω .)

The two bending processes mentioned in the last section preserve $\mathcal{S}_f$: If u_1, u_2 belong to $\mathcal{S}_f$, then so does $\max\{u_1, u_2\}$; if $u \in \mathcal{S}_f$ and K is a closed ball in Ω , then the Poisson modification of u for K belongs to $\mathcal{S}_f$.

Perron's candidate solution for the Dirichlet problem with boundary data f is the function defined on $\bar{\Omega}$ by

$$\mathcal{P}[f](x) = \sup\{u(x) : u \in \mathcal{S}_f\}.$$

We call $\mathcal{P}[f]$ the *Perron function* for f . Note that $m \leq \mathcal{P}[f] \leq M$ on $\bar{\Omega}$, where m and M are the minimum and maximum values of f on $\partial\Omega$. Note also that $\mathcal{P}[f] \leq f$ on $\partial\Omega$.

To motivate the definition of $\mathcal{P}[f]$, suppose that v solves the Dirichlet problem on Ω with boundary data f . Then $v \in \mathcal{S}_f$, so that $v \leq \mathcal{P}[f]$ on $\bar{\Omega}$. On the other hand, Theorem 11.3 shows that every function in $\mathcal{S}_f$ is bounded above on $\bar{\Omega}$ by v , so that $\mathcal{P}[f] \leq v$ on $\bar{\Omega}$. In other words, if there is a solution, it must be $\mathcal{P}[f]$.

Remarkably, even though $\mathcal{P}[f]$ may not be a solution, it is always harmonic.

11.6 Theorem: $\mathcal{P}[f]$ is harmonic on Ω .

PROOF: Let $\bar{B}(a, r) \subset \Omega$. It suffices to show that $\mathcal{P}[f]$ is harmonic on $B(a, r)$.

Choose a sequence (u_k) in $\mathcal{S}_f$ such that $u_k(a) \rightarrow \mathcal{P}[f](a)$. Replacing u_k by the Poisson modification of $\max\{u_1, \dots, u_k\}$ for $\bar{B}(a, r)$, we obtain a sequence in $\mathcal{S}_f$ that increases on Ω and whose terms are harmonic on $B(a, r)$. Denoting this new sequence by (u_k) as well, we still have $u_k(a) \rightarrow \mathcal{P}[f](a) < \infty$. By Harnack's Principle (3.8), (u_k) converges uniformly on compact subsets of $B(a, r)$ to a function u harmonic on $B(a, r)$. The proof will be completed by showing $\mathcal{P}[f] = u$ on $B(a, r)$.

We clearly have $u \leq \mathcal{P}[f]$ on $B(a, r)$. To prove the reverse inequality, let $v \in \mathcal{S}_f$; we need to show that $v \leq u$ on $B(a, r)$. Let v_k denote the Poisson modification of $\max\{u_k, v\}$ for $\bar{B}(a, r)$. Because $u(a) = \mathcal{P}[f](a)$ and $v_k \in \mathcal{S}_f$, we have $v_k(a) \leq u(a)$ for all k . Furthermore, v_k is harmonic on $B(a, r)$ and $\max\{u_k, v\} \leq v_k$ on $\bar{B}(a, r)$ by subharmonicity. Thus for positive $s < r$,

$$\begin{aligned} u(a) &\geq v_k(a) \\ &= \int_S v_k(a + s\zeta) d\sigma(\zeta) \\ &\geq \int_S [\max\{u_k, v\}](a + s\zeta) d\sigma(\zeta). \end{aligned}$$

Letting $k \rightarrow \infty$, we see that the mean-value property of u on $B(a, r)$ gives

$$\int_S u(a + s\zeta) d\sigma(\zeta) \geq \int_S [\max\{u, v\}](a + s\zeta) d\sigma(\zeta).$$

It follows that $v \leq u$ on $B(a, r)$, proving that $\mathcal{P}[f] \leq u$ on $B(a, r)$, as desired. $\square$

Barrier Functions and Geometric Criteria for Solvability

Let $\zeta \in \partial\Omega$. We call a continuous real-valued function u on $\bar{\Omega}$ a *barrier function* for Ω at ζ provided that u is subharmonic on Ω , $u < 0$ on $\bar{\Omega} \setminus \{\zeta\}$, and $u(\zeta) = 0$. When such a u exists, we say that Ω has a *barrier* at ζ . (After Theorems 11.11 and 11.16, the reader may concede that the term “barrier” is apt. Poincaré introduced barrier

functions into the study of the Dirichlet problem; Lebesgue coined the term “barrier” and generalized the notion.)

11.7 Theorem: *If Ω has a barrier at $\zeta \in \partial\Omega$, then $\mathcal{P}[f](x) \rightarrow f(\zeta)$ as $x \rightarrow \zeta$ within $\bar{\Omega}$.*

PROOF: Suppose u is a barrier function for Ω at $\zeta \in \partial\Omega$. Let $\varepsilon > 0$. By the continuity of f on $\partial\Omega$, we may choose a ball $B(\zeta, r)$ such that $f(\zeta) - \varepsilon < f < f(\zeta) + \varepsilon$ on $\partial\Omega \cap B(\zeta, r)$. Because u is negative and continuous on the compact set $\bar{\Omega} \setminus B(\zeta, r)$, there exists a positive constant C such that

$$11.8 \quad f(\zeta) - \varepsilon + Cu < f < f(\zeta) + \varepsilon - Cu$$

on $\partial\Omega \setminus B(\zeta, r)$. The nonpositivity of u then shows that 11.8 is valid everywhere on $\partial\Omega$.

We claim that

$$11.9 \quad f(\zeta) - \varepsilon + Cu \leq \mathcal{P}[f] \leq f(\zeta) + \varepsilon - Cu$$

on $\bar{\Omega}$. The first inequality in 11.9 holds because $f(\zeta) - \varepsilon + Cu \in \mathcal{S}_f$. For the other inequality, let $v \in \mathcal{S}_f$. Then $v \leq f$ on $\partial\Omega$, and therefore $v + Cu < f(\zeta) + \varepsilon$ on $\partial\Omega$ by 11.8. Theorem 11.3 then shows that $v + Cu < f(\zeta) + \varepsilon$ on $\bar{\Omega}$, from which the inequality on the right of 11.9 follows.

Because $u(\zeta) = 0$ and ε is arbitrary, the continuity of u and 11.9 give us the desired convergence of $\mathcal{P}[f](x)$ to $f(\zeta)$ as $x \rightarrow \zeta$ within $\bar{\Omega}$. $\square$

11.10 Theorem: *The Dirichlet problem for bounded Ω is solvable if and only if Ω has a barrier at each $\zeta \in \partial\Omega$.*

PROOF: Suppose that the Dirichlet problem for Ω is solvable and $\zeta \in \partial\Omega$. The function f defined on $\partial\Omega$ by $f(x) = -|x - \zeta|$ is continuous on $\partial\Omega$; the solution to the Dirichlet problem on Ω with boundary data f is then a barrier function for Ω at ζ .

Conversely, suppose that each point of $\partial\Omega$ has a barrier function. Theorems 11.6 and 11.7 then show that $\mathcal{P}[f]$ solves the Dirichlet problem with boundary data f whenever f is continuous and real valued on $\partial\Omega$. $\square$

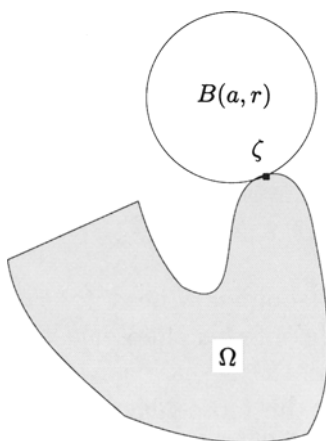
Theorem 11.10 reduces the Dirichlet problem to a local boundary behavior question that we call the *barrier problem*. Have we made progress, or have we merely traded one boundary behavior problem for another? We will see that the barrier problem can be solved in many cases of interest. For example, the next result will enable us to prove that the Dirichlet problem is solvable for any bounded convex domain.

11.11 External Ball Condition: *If $\zeta \in \partial\Omega$ belongs to a closed ball contained in the complement of Ω , then Ω has a barrier at ζ .*

PROOF: Suppose $\bar{B}(a, r)$ is a closed ball in the complement of Ω such that $\partial B(a, r) \cap \partial\Omega = \{\zeta\}$. Define u on $\mathbf{R}^n \setminus \{a\}$ by

$$u(x) = \begin{cases} \log r - \log |x - a| & \text{if } n = 2 \\ |x - a|^{2-n} - r^{2-n} & \text{if } n > 2. \end{cases}$$

Then u is a barrier for Ω at ζ . □



The external ball condition.

Consider now the case where Ω is bounded and convex. Each $\zeta \in \partial\Omega$ then belongs to a closed half-space contained in the complement of Ω , which implies that the external ball condition is satisfied at each $\zeta \in \partial\Omega$. By 11.11, Ω has a barrier at every point in its boundary. The following corollary is thus a consequence of Theorem 11.10.

11.12 Corollary: *If Ω is bounded and convex, then the Dirichlet problem for Ω is solvable.*

Theorem 11.11 also enables us to prove that the Dirichlet problem is solvable for bounded Ω when $\partial\Omega$ is sufficiently smooth. To make this more precise, let us say that Ω has C^k -boundary if for every $\zeta \in \partial\Omega$ there exists a neighborhood ω of ζ and a real-valued $\varphi \in C^k(\omega)$ satisfying the following conditions:

- (a) $\Omega \cap \omega = \{x \in \omega : \varphi(x) < 0\}$;
- (b) $\partial\Omega \cap \omega = \{x \in \omega : \varphi(x) = 0\}$;
- (c) $\nabla\varphi(\zeta) \neq 0$ for every $\zeta \in \partial\Omega \cap \omega$.

Here k is any positive integer. The function φ is called a *local defining function* for Ω .

Assume now that Ω has C^2 -boundary. For simplicity, suppose $0 \in \partial\Omega$ and that the tangent space of $\partial\Omega$ at 0 is $\mathbf{R}^{n-1} \times \{0\}$. Then near 0, $\partial\Omega$ is the graph of a C^2 -function ψ defined near $0 \in \mathbf{R}^{n-1}$, where $\nabla\psi(0) = 0$; this follows easily from the implicit function theorem. Because $\nabla\psi(0) = 0$, $|\psi(x)| = O(|x|^2)$ as $x \rightarrow 0$ by Taylor's Theorem. This implies (we leave the details to the reader) that Ω satisfies the external ball condition at 0. The preceding argument, after a translation and rotation, applies to any boundary point of Ω . From Theorems 11.10 and 11.11 we thus obtain the following result:

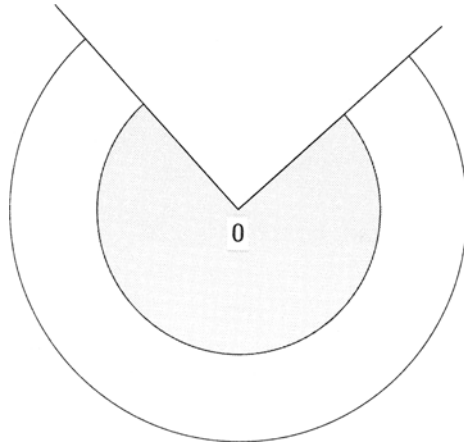
11.13 Corollary: *If Ω is bounded and has C^2 -boundary, then the Dirichlet problem for Ω is solvable.*

A domain with C^1 -boundary need not satisfy the external ball condition (Exercise 15(a) of this chapter). We now take up a condition on $\partial\Omega$ that covers the C^1 -case as well as many “nonsmooth” cases. The prototype for this more-general Ω is the domain $B \setminus \bar{\Gamma}_\alpha(0)$, where $\Gamma_\alpha(0)$ is the cone defined in Chapter 2; see the following diagram, where $B \setminus \bar{\Gamma}_\alpha(0)$ and one of its dilates are pictured.

We will need the following maximum principle for $B \setminus \bar{\Gamma}_\alpha(0)$.

11.14 Lemma: *Let $\Omega = B \setminus \bar{\Gamma}_\alpha(0)$. Suppose u is real valued and continuous on $\bar{\Omega} \setminus \{0\}$, u is bounded and harmonic on Ω , and $u \leq M$ on $\partial\Omega \setminus \{0\}$. Then $u \leq M$ on Ω .*

PROOF: If $n > 2$, apply the maximum principle (Corollary 1.6) to the function $u(x) - M - \varepsilon|x|^{2-n}$ on Ω and let $\varepsilon \rightarrow 0$. When $n = 2$, the same argument applies with $\log 1/|x|$ in place of $|x|^{2-n}$. $\square$



$B \setminus \bar{\Gamma}_\alpha(0)$ and one of its dilates.

With Ω as in Lemma 11.14, note that $r\Omega = (rB) \cap \Omega$ for any $r \in (0, 1)$. This fact will be crucial in proving the next result.

11.15 Lemma: Let $\Omega = B \setminus \bar{\Gamma}_\alpha(0)$. On $\partial\Omega$, set $f(x) = |x|$, and on $\bar{\Omega}$, set $u = \mathcal{P}[f]$. Then $-u$ is a barrier for Ω at 0.

PROOF: The function u is harmonic on Ω , with $0 \leq u \leq 1$ on $\bar{\Omega}$. Because the external ball condition holds at each point of $\partial\Omega \setminus \{0\}$, u is continuous and positive on $\bar{\Omega} \setminus \{0\}$. The proof will be completed by showing that $\limsup_{x \rightarrow 0} u(x) = 0$.

Fix $r \in (0, 1)$. By the maximum principle, there exists a constant $c < 1$ such that $u \leq c$ on $\partial(r\Omega) \setminus \{0\}$. Now define

$$v(x) = u(x) - \max\{r, c\}u(x/r)$$

for $x \in r\bar{\Omega}$. It is easy to check that $v \leq 0$ on $\partial(r\Omega) \setminus \{0\}$. Applying Lemma 11.14, we obtain $v \leq 0$ on $r\Omega$. Thus

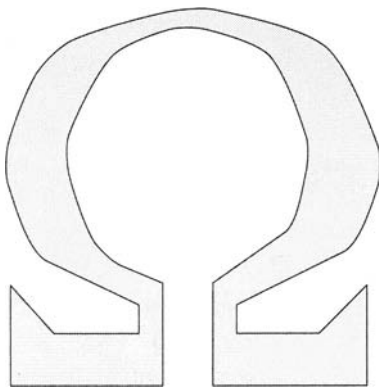
$$\limsup_{x \rightarrow 0} u(x) \leq \max\{r, c\} \limsup_{x \rightarrow 0} u(x/r) = \max\{r, c\} \limsup_{x \rightarrow 0} u(x).$$

Because $\max\{r, c\} < 1$, the left side of the inequality above is 0. $\square$

The next result shows that the Dirichlet problem is solvable in any bounded domain satisfying the “external cone condition” at each of its boundary points. By a cone we shall mean any set of the form $a + T(\Gamma_\alpha^h(0))$, where $a \in \mathbf{R}^n$, T is a rotation, and $\Gamma_\alpha^h(0)$ is the truncation of $\Gamma_\alpha(0)$ defined in Chapter 7. We refer to a as the vertex of such a cone.

11.16 External Cone Condition: *If Ω is bounded and $\zeta \in \partial\Omega$ is the vertex of a cone contained in the complement of Ω , then Ω has a barrier at ζ .*

PROOF: Subharmonic functions are preserved by translations, dilations, and rotations, so we may assume that $\zeta = 0$ and that $\Omega \cap B \subset B \setminus \bar{\Gamma}_\alpha(0)$ for some $\alpha > 0$. Consider now the function $-u$ obtained in Lemma 11.15. This function is identically -1 on $S \setminus \bar{\Gamma}_\alpha(0)$. Thus if we extend this function by defining it to equal -1 on $\mathbf{R}^n \setminus B$, the result is a barrier for Ω at 0. $\square$



An Ω satisfying the external cone condition.

Non-extendability Results

We now turn from the Dirichlet problem to another kind of boundary behavior question. Recall that if $\Omega \subset \mathbf{R}^2 = \mathbf{C}$, then there exists

a function holomorphic on Ω that does not extend holomorphically to any larger set. This is usually proved by first showing that each discrete subset of Ω is the precise zero set of some function holomorphic on Ω . Because a discrete subset of Ω that clusters at each point of $\partial\Omega$ can always be chosen, some holomorphic function on Ω is non-extendable.

Remarkably, there exist domains in $\mathbf{C}^m$, $m > 1$, for which every holomorphic function extends across the boundary. For example, every holomorphic function on $\mathbf{C}^2 \setminus \{0\}$ extends to be entire on $\mathbf{C}^2$; see [6], pages 5–6, for details.

What about harmonic functions of more than two real variables? The next result shows that given any $\Omega \subset \mathbf{R}^n$, $n > 2$, a non-extendable positive harmonic function on Ω can always be produced.

11.17 Theorem: *Suppose $n > 2$ and $\Omega \subset \mathbf{R}^n$. Then there exists a positive harmonic function u on Ω such that*

$$\limsup_{x \rightarrow \zeta} u(x) = \infty$$

for every $\zeta \in \partial\Omega$.

PROOF: Assume first that Ω is connected. Let $\{\zeta_1, \zeta_2, \dots\}$ be a countable dense subset of $\partial\Omega$. Fixing $a \in \Omega$, we may choose positive constants c_m such that

$$c_m |a - \zeta_m|^{2-n} < 2^{-m}$$

for $m = 1, 2, \dots$. For $x \in \Omega$, define

$$u(x) = \sum_{m=1}^{\infty} c_m |x - \zeta_m|^{2-n}.$$

Each term in this series is positive and harmonic on Ω , and the series converges at $a \in \Omega$. By Harnack's Principle, u is harmonic on Ω , and we easily verify that u satisfies the conclusion of the theorem.

If Ω is not connected, we apply the preceding to each connected component of Ω to produce the desired function. $\square$

If $\Omega \subset \mathbf{R}^2$, it may not be possible to construct a positive harmonic function on Ω that is unbounded near every point of $\partial\Omega$.

Recall, for example, that every positive harmonic function on $\mathbf{R}^2 \setminus \{0\}$ is constant (Corollary 3.3). Our next result, however, is valid for all $n \geq 2$.

11.18 Theorem: *Let $\Omega \subset \mathbf{R}^n$. Then there exists a real-valued harmonic function u on Ω such that*

$$\liminf_{x \rightarrow \zeta} u(x) = -\infty, \quad \limsup_{x \rightarrow \zeta} u(x) = \infty$$

for every $\zeta \in \partial\Omega$.

PROOF: In the proof we will assume that $n > 2$; we leave the $n = 2$ case as an exercise.

Let I denote the set of isolated points of $\partial\Omega$. We assume that $\partial\Omega \setminus \bar{I}$ is non-empty; the proof that follows will easily adapt to the case $\partial\Omega = \bar{I}$. Select disjoint countable dense subsets D_- and D_+ of $\partial\Omega \setminus \bar{I}$, and write

$$D_- \cup D_+ \cup I = \{\zeta_1, \zeta_2, \dots\}.$$

Now choose pairwise disjoint compact sets $E_1, E_2, \dots$ such that for each m ,

- (a) $E_m \subset \Omega \cup \{\zeta_m\}$;
- (b) ζ_m is a limit point of E_m .

For $\zeta_m \in I$ we insist that E_m be a closed ball of positive radius centered at ζ_m . For $\zeta_m \in D_- \cup D_+$ we will not be as fussy; for example, we can take E_m to be the union of $\{\zeta_m\}$ with a sequence in Ω converging to ζ_m . The pairwise disjointness of $\{E_m\}$ is easy to arrange by induction.

Set $v(x) = |x|^{2-n}$, $w(x) = x_1|x|^{-n}$, and for $m = 1, 2, \dots$, define

$$u_m(x) = \begin{cases} -v(x - \zeta_m) & \text{if } \zeta_m \in D_- \\ v(x - \zeta_m) & \text{if } \zeta_m \in D_+ \\ w(x - \zeta_m) & \text{if } \zeta_m \in I. \end{cases}$$

Note that u_m is harmonic on $\mathbf{R}^n \setminus \{\zeta_m\}$.

Choose compact sets $K_1, K_2, \dots \subset \Omega$ such that

$$K_1 \subset \text{int } K_2 \subset K_2 \subset \dots$$

and $\Omega = \cup K_m$. We may then choose positive constants c_m such that

$$c_m |u_m| \leq 2^{-m}$$

on $K_m \cup E_1 \cup \cdots \cup E_{m-1}$. (Let E_0 be the empty set.)

Finally, define

$$u = \sum_{m=1}^{\infty} c_m u_m.$$

Because this series converges uniformly on compact subsets of Ω , u is harmonic on Ω .

To check the boundary behavior of u , we first consider the case where $\zeta \in \partial\Omega \setminus \bar{I}$. Let $\varepsilon > 0$. Then $B(\zeta, \varepsilon)$ contains some $\zeta_m \in D_-$. On the corresponding E_m , the series $\sum_{j \neq m} c_j u_j$ converges uniformly to a function continuous on E_m . Because $c_m u_m(x) \rightarrow -\infty$ as $x \rightarrow \zeta_m$ within E_m , the infimum of u over $B(\zeta, \varepsilon) \cap \Omega$ is $-\infty$. Similarly, the supremum of u over $B(\zeta, \varepsilon) \cap \Omega$ is $+\infty$, giving us the desired conclusion.

Now suppose that $\zeta \in \bar{I}$. If $\varepsilon > 0$, then $B(\zeta, \varepsilon)$ contains some $\zeta_m \in I$. The series $\sum_{j \neq m} c_j u_j$ then converges uniformly to a continuous function on the closed ball E_m . Because $c_m u_m$ in this case maps any punctured ball $B(\zeta_m, r) \setminus \{\zeta_m\}$ onto $\mathbf{R}$, we are done. $\square$

If Ω is locally connected near $\partial\Omega$ (for example, if Ω is convex or has C^1 -boundary), then the function u of Theorem 11.18 satisfies $u(B(\zeta, \varepsilon) \cap \Omega) = \mathbf{R}$ for every $\zeta \in \partial\Omega$ and every $\varepsilon > 0$. (See also Exercise 22 of this chapter.)

Exercises

1. Show that every translation, dilation, and rotation of a subharmonic function is subharmonic.
2. Suppose that u is subharmonic on Ω and that φ is increasing and convex on an open interval containing $u(\Omega)$. Prove that $\varphi \circ u$ is subharmonic on Ω .
3. Let u be a real-valued continuous function on Ω . Show that u is subharmonic on Ω if and only if for every compact $K \subset \Omega$ the following holds: If $u \leq v$ on ∂K , where v is continuous on K and harmonic on the interior of K , then $u \leq v$ on K .
4. Suppose that u is subharmonic on Ω and $a \in \Omega$. Show that the function $r \mapsto \int_S u(a + r\zeta) d\sigma(\zeta)$ is increasing. Conclude that the submean-value inequality 11.1 is valid for all $r < d(a, \partial\Omega)$.
5. Show that if a sequence of functions subharmonic on Ω converges uniformly on compact subsets of Ω , then the limit function is subharmonic on Ω .
6. Show that $|u|^p$ is subharmonic on Ω whenever u is harmonic on Ω and $1 \leq p < \infty$. (This and Exercise 4 imply that $\|u_r\|_p$ increases to $\|u\|_{h^p}$ as $r \rightarrow 1$ for every $u \in h^p(B)$.)
7. Suppose $u \in C^2(\Omega)$ is real valued. Use Taylor's Theorem to show that u is subharmonic on Ω if and only if $\Delta u \geq 0$ on Ω . (*Hint:* To show $\Delta u \geq 0$ implies u is subharmonic, first assume $\Delta u > 0$. Then consider the functions $u(x) + \varepsilon|x|^2$.)
8. Show that $|x|^p$ is subharmonic on $\mathbf{R}^n$ for every $p > 0$.
9. (a) Show that a subharmonic function on $\mathbf{R}^2$ that is bounded above must be constant.
(b) Suppose $n > 2$. Find a nonconstant subharmonic function on $\mathbf{R}^n$ that is bounded above.

10. Assume that f is holomorphic on $\Omega \subset \mathbf{R}^2 = \mathbf{C}$ and that u is subharmonic on $f(\Omega)$. Prove that $u \circ f$ is subharmonic on Ω .
11. Give an easier proof of Theorem 11.16 for the case $n = 2$.
12. With $n = 3$, let Ω denote the open unit ball with the line segment from $\mathbf{S}$ to $\mathbf{N}$ removed. Show that the Dirichlet problem for Ω is not solvable.
13. Show that when $n = 2$, the Dirichlet problem is solvable for bounded open sets satisfying an “external segment condition”. (*Hint*: An appropriate conformal map of $B_2 \setminus \{0\}$ onto the plane minus a line segment may be useful.)
14. Show that the Dirichlet problem is solvable for $B_3 \setminus (\bar{H}_2 \times \{0\})$.
15. (a) Give an example of a bounded Ω with C^1 -boundary such that the external ball condition fails for some $\zeta \in \partial\Omega$.
(b) Show that a domain with C^1 -boundary satisfies the external cone condition at each of its boundary points.
16. Suppose Ω is bounded. Show that the Dirichlet problem for Ω is solvable if and only if $\mathcal{P}[-f] = -\mathcal{P}[f]$ on $\bar{\Omega}$ for every real-valued continuous f on $\partial\Omega$.
17. Suppose that Ω is bounded and $a \in \Omega$. Show that the map $f \mapsto \mathcal{P}[f](a)$ is a bounded real-linear functional on the space of real-valued continuous functions on $\partial\Omega$.
18. Suppose Ω is bounded and $a \in \Omega$. Show that there exists a unique positive Borel measure μ_a on $\partial\Omega$, with $\mu_a(\partial\Omega) = 1$, such that

$$\mathcal{P}[f](a) = \int_{\partial\Omega} f \, d\mu_a$$
 for every real-valued continuous f on $\partial\Omega$. (The measure μ_a is called *harmonic measure* for Ω at a .)
19. Prove Theorem 11.17 in the case $n = 2$.
20. Show that if $\Omega \subset \mathbf{R}^2$ is bounded, then there exists a positive harmonic function u on Ω such that $\limsup_{x \rightarrow \zeta} u(x) = \infty$ for every $\zeta \in \partial\Omega$.

21. Suppose $\Omega \subset \mathbf{C}$ and $\{\zeta_1, \zeta_2, \dots\}$ is a countable dense subset of $\partial\Omega$. Construct a holomorphic function on Ω of the form

$$\sum_{m=1}^{\infty} \frac{c_m}{z - \zeta_m}$$

that does not extend across any part of $\partial\Omega$. (This should be easier than the proof of Theorem 11.18.)

22. Given an arbitrary $\Omega \subset \mathbf{R}^n$, does there always exist a real-valued harmonic u on Ω such that $u(B(\zeta, \varepsilon) \cap \Omega) = \mathbf{R}$ for every $\zeta \in \partial\Omega$ and every $\varepsilon > 0$?

APPENDIX A

Volume, Surface Area, and Integration on Spheres

Volume of the Ball and Surface Area of the Sphere

In this section we compute the volume of the unit ball and surface area of the unit sphere in $\mathbf{R}^n$. Recall that $B = B_n$ denotes the unit ball in $\mathbf{R}^n$ and that $V = V_n$ denotes volume measure in $\mathbf{R}^n$. We begin by evaluating the constant $V(B)$, which appears in several formulas throughout the book.

A.1 Proposition: *The volume of the unit ball in $\mathbf{R}^n$ equals*

$$\begin{aligned} & \frac{\pi^{n/2}}{(n/2)!} && \text{if } n \text{ is even,} \\ & \frac{2^{(n+1)/2} \pi^{(n-1)/2}}{1 \cdot 3 \cdot 5 \cdots n} && \text{if } n \text{ is odd.} \end{aligned}$$

PROOF: Assume $n > 2$, denote a typical point in $\mathbf{R}^n$ by (x, y) , where $x \in \mathbf{R}^2$ and $y \in \mathbf{R}^{n-2}$, and express the volume $V_n(B_n)$ as an iterated integral:

$$\begin{aligned}
V_n(B_n) &= \int_{B_n} 1 \, dV_n \\
&= \int_{B_2} \int_{(1-|x|^2)^{1/2} B_{n-2}} 1 \, dV_{n-2}(y) \, dV_2(x).
\end{aligned}$$

The inner integral on the last line equals the $(n-2)$ -dimensional volume of a ball in $\mathbf{R}^{n-2}$ with radius $(1-|x|^2)^{1/2}$. Thus

$$V_n(B_n) = V_{n-2}(B_{n-2}) \int_{B_2} (1-|x|^2)^{(n-2)/2} \, dV_2(x).$$

Switching to the usual polar coordinates in $\mathbf{R}^2$, we get

$$\begin{aligned}
V_n(B_n) &= V_{n-2}(B_{n-2}) \int_{-\pi}^{\pi} \int_0^1 (1-r^2)^{(n-2)/2} r \, dr \, d\theta \\
&= \frac{2\pi}{n} V_{n-2}(B_{n-2}).
\end{aligned}$$

The last formula can be easily used to prove the desired formula for $V_n(B_n)$ by induction in steps of 2, starting with $V_2(B_2) = \pi$ and $V_3(B_3) = 4\pi/3$. $\square$

Readers familiar with the gamma function should be able to rewrite the formula given by A.1 as a single expression that holds whether n is even or odd (see Exercise 6 of this appendix).

Turning now to surface-area measure, we let S_n denote the unit sphere in $\mathbf{R}^n$.^{*} Unnormalized surface-area measure on S_n will be denoted by s_n and normalized surface-area measure on S_n will be denoted by σ_n . We presume some familiarity with surface-area measure for smooth manifolds. The arguments we give in the remainder of this appendix will be more intuitive than rigorous; the reader should have no trouble filling in the missing details.

Let us find the relationship between $V_n(B_n)$ and $s_n(S_n)$. We do this with an old trick from calculus: For $h \approx 0$ we have

$$\begin{aligned}
((1+h)^n - 1)V_n(B_n) &= V_n((1+h)B_n) - V_n(B_n) \\
&\approx s_n(S_n)h.
\end{aligned}$$

^{*}A more common notation is S^{n-1} , which emphasizes that the sphere has dimension $n-1$ as a manifold. We use S_n to emphasize that the sphere lives in $\mathbf{R}^n$.

Dividing by h and letting $h \rightarrow 0$, we obtain $nV_n(B_n) = s_n(S_n)$. We record this result in the following proposition.

A.2 Proposition: *The unnormalized surface area of the unit sphere in $\mathbf{R}^n$ equals $nV_n(B_n)$.*

Slice Integration on Spheres

The map $\psi: B_{n-1} \rightarrow S_n$ defined by

$$\psi(x) = (x, \sqrt{1 - |x|^2})$$

parameterizes the upper hemisphere of S_n . The corresponding change of variables is given by the formula

$$\text{A.3} \quad ds_n(\psi(x)) = \frac{dV_{n-1}(x)}{\sqrt{1 - |x|^2}}.$$

Equation A.3 is found in most calculus texts in the cases $n = 2, 3$.

Consider now the map $\Psi: B_{n-k} \times S_k \rightarrow S_n$ defined by

$$\Psi(x, \zeta) = (x, \sqrt{1 - |x|^2} \zeta).$$

Here $1 \leq k < n$. The map Ψ is one-to-one, and the range of Ψ is S_n minus a set that has s_n -measure 0 (namely, the set of points on S_n whose last k coordinates vanish). We wish to find the change of variables formula associated with Ψ .

Observe that $B_{n-k} \times S_k$ is an $(n - 1)$ -dimensional submanifold of $\mathbf{R}^n$ whose element of surface area is $d(V_{n-k} \times s_k)$. For fixed x , Ψ changes $(k - 1)$ -dimensional area on $\{x\} \times S_k$ by the factor $(1 - |x|^2)^{(k-1)/2}$. For fixed ζ , A.3 shows that Ψ changes $(n - k)$ -dimensional area on $B_{n-k} \times \{\zeta\}$ by the factor $(1 - |x|^2)^{-1/2}$. Furthermore, the submanifolds $\Psi(\{x\} \times S_k)$ and $\Psi(B_{n-k} \times \{\zeta\})$ are perpendicular at their point of intersection, as is easily checked. The last statement implies that Ψ changes $(n - 1)$ -dimensional measure on $B_{n-k} \times S_k$ by the product of the factors above. In other words,

$$\text{A.4} \quad ds_n(\Psi(x, \zeta)) = (1 - |x|^2)^{(k-2)/2} dV_{n-k}(x) ds_k(\zeta).$$

The last equation leads to a useful formula (A.5) for integrating over a sphere by iterating an integral over lower-dimensional

spherical slices. This formula is an immediate consequence of A.2 and A.4. (We state A.5 in terms of normalized surface-area measure because that is what we have used most often.)

A.5 Theorem: *Let f be a Borel measurable, integrable function on S_n . If $1 \leq k < n$, then $\int_{S_n} f d\sigma_n$ equals*

$$\frac{k}{n} \frac{V(B_k)}{V(B_n)} \int_{B_{n-k}} (1 - |x|^2)^{\frac{k-2}{2}} \int_{S_k} f(x, \sqrt{1 - |x|^2} \zeta) d\sigma_k(\zeta) dV_{n-k}(x).$$

Slice Integration: Special Cases

Some cases of Theorem A.5 deserve special mention. We begin by choosing $k = n - 1$, which is the largest permissible value of k . This corresponds to decomposing S_n into spheres of one less dimension by intersecting S_n with the family of hyperplanes orthogonal to the first coordinate axis. The ball B_{n-k} is just the unit interval $(-1, 1)$, and so for $x \in B_{n-k}$, we can write x^2 instead of $|x|^2$. Thus we obtain the following corollary of A.5.

A.6 Corollary: *Let f be a Borel measurable, integrable function on S_n . Then $\int_{S_n} f d\sigma_n$ equals*

$$\frac{n-1}{n} \frac{V(B_{n-1})}{V(B_n)} \int_{-1}^1 (1 - x^2)^{\frac{n-3}{2}} \int_{S_{n-1}} f(x, \sqrt{1 - x^2} \zeta) d\sigma_{n-1}(\zeta) dx.$$

At the other extreme we can choose $k = 1$. This corresponds to decomposing S_n into pairs of points by intersecting S_n with the family of lines parallel to the n^{th} coordinate axis. The sphere S_1 is the two-point set $\{-1, 1\}$, and $d\sigma_1$ is counting measure on this set, normalized so that each point has measure $1/2$. Thus we obtain the following corollary of A.5.

A.7 Corollary: *Let f be a Borel measurable, integrable function on S_n . Then $\int_{S_n} f d\sigma_n$ equals*

$$\frac{1}{nV(B_n)} \int_{B_{n-1}} \frac{f(x, \sqrt{1 - |x|^2}) + f(x, -\sqrt{1 - |x|^2})}{\sqrt{1 - |x|^2}} dV_{n-1}(x).$$

Let us now try $k = 2$ (assuming $n > 2$). Thus in A.5 the term $(1 - |x|^2)^{(k-2)/2}$ disappears. The variable ζ in the formula given by A.5 now ranges over the unit circle in $\mathbf{R}^2$, so we can replace ζ by $(\cos \theta, \sin \theta)$, which makes $d\sigma_2(\zeta)$ equal to $d\theta/(2\pi)$. Thus we obtain the following corollary of A.5.

A.8 Corollary ($n > 2$): *Let f be a Borel measurable, integrable function on S_n . Then $\int_{S_n} f d\sigma_n$ equals*

$$\frac{1}{nV(B_n)} \int_{B_{n-2}} \int_{-\pi}^{\pi} f(x, \sqrt{1 - |x|^2} \cos \theta, \sqrt{1 - |x|^2} \sin \theta) d\theta dV_{n-2}(x).$$

An important special case of the last result occurs when $n = 3$. In this case B_{n-2} is just the interval $(-1, 1)$, and we get the following corollary.

A.9 Corollary: *Let f be a Borel measurable, integrable function on S_3 . Then $\int_{S_3} f d\sigma_3$ equals*

$$\frac{1}{4\pi} \int_{-1}^1 \int_{-\pi}^{\pi} f(x, \sqrt{1 - x^2} \cos \theta, \sqrt{1 - x^2} \sin \theta) d\theta dx.$$

Exercises

1. Prove that

$$\int_{\mathbf{R}^n} \frac{1}{(|x| + 1)^p} dV(x) < \infty$$

if and only if $p > n$.

2. (a) Consider the region on the unit sphere in $\mathbf{R}^3$ lying between two parallel planes that intersect the sphere. Show that the area of this region depends only on the distance between the two planes. (This result was discovered by the ancient Greeks.)
 (b) Show that the result in part (a) does not hold in $\mathbf{R}^n$ if $n \neq 3$ and “planes” are replaced by “hyperplanes”.
3. Let f be a Borel measurable, integrable function on the unit sphere S_4 in $\mathbf{R}^4$. Define a function Ψ mapping the rectangular box $[-1, 1] \times [-1, 1] \times [-\pi, \pi]$ to S_4 by setting $\Psi(x, y, \theta)$ equal to

$$(x, \sqrt{1-x^2}y, \sqrt{1-x^2}\sqrt{1-y^2}\cos\theta, \sqrt{1-x^2}\sqrt{1-y^2}\sin\theta).$$

Prove that

$$\int_{S_4} f d\sigma_4 = \frac{1}{2\pi^2} \int_{-1}^1 \sqrt{1-x^2} \int_{-1}^1 \int_{-\pi}^{\pi} f(\Psi(x, y, \theta)) d\theta dy dx.$$

4. Without writing down anything or using a computer, evaluate

$$\int_{S_5} \zeta_1^2 d\sigma(\zeta).$$

5. Let m be a positive integer. Use A.6 to give an explicit formula for

$$\int_{-1}^1 (1-x^2)^{m/2} dx.$$

6. For readers familiar with the gamma function Γ : Prove that the volume of the unit ball in $\mathbf{R}^n$ equals

$$\frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)}.$$

APPENDIX B

Mathematica and Harmonic Function Theory

Using *Mathematica*,* a symbolic processing program, the authors have written routines to manipulate many of the expressions that arise in the study of harmonic functions. These routines allow the user to make symbolic calculations that would take a prohibitive amount of time if done without a computer. For example, Poisson integrals of polynomials can be computed exactly.

Our routines for symbolic manipulation of harmonic functions are distributed free of charge by electronic mail. They are designed to work on any computer that runs *Mathematica*. Requests for our software package should be sent to axler@math.msu.edu. Comments, suggestions, and bug reports should also be sent to the same electronic address.

**Mathematica* is a registered trademark of Wolfram Research.

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